A Numerical Verification Method of Solutions for the Navier-Stokes Equations

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Incompressible Navier-Stokes Equations

$$\begin{cases} -\nu\Delta u + \nabla p = -(u\cdot\nabla)u + f & \text{in} \quad \Omega, \\ \operatorname{div} u = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Ω : convex polygon $\subset \mathbb{R}^2$

$u(x,y)$: velocity field, $u = (u_1, u_2)^T$

$p(x,y)$: kinematic pressure field

f : density of body forces, $f = (f_1, f_2)^T$

ν : viscosity constant, $\nu > 0$

$Re = 1/\nu$: Reynolds number

$$(u\cdot\nabla)u = \sum_{i=1}^{n} u_i \frac{\partial u}{\partial x_i}$$

$$\operatorname{div} u = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i}$$

Weak Formulation

find $[u,p] \in H_0^1(\Omega)^2 \times L_0^2(\Omega)$ such that

$$\begin{cases} \nu(\nabla u, \nabla v) - (p, \operatorname{div} v) = -((u\cdot\nabla)u, v) + (f, v) & \forall v \in H_0^1(\Omega)^2, \\ -(\operatorname{div} u, q) = 0 & \forall q \in L_0^2(\Omega). \end{cases}$$

- $L^2(\Omega)$: set of square-integrable functions on Ω.

$$|u|_0 \equiv \left(\int_\Omega u^2\, dxdy\right)^{1/2}, \qquad (u, v) \equiv \int_\Omega uv\, dxdy.$$

- $L_0^2(\Omega) \equiv \{v \in L^2(\Omega)\ ;\ \int_\Omega v\, dx\, dy = 0\}$.
- $H^k(\Omega)$: k-th order L^2-Sobolev space on Ω.
- $H_0^1(\Omega) \equiv \{v \in H^1(\Omega)\ ;\ v = 0 \text{ on } \partial\Omega\}$.

$$|u|_1 \equiv |\nabla u|_0.$$

- $H^{-1}(\Omega)^2$: dual space of $H_0^1(\Omega)^2$.

$$|f|_{-1} \equiv \sup_{v \in H_0^1(\Omega)^2} \frac{< f, v >}{|v|_1},$$

where $< \cdot, \cdot >$ is a duality pairing between $H^{-1}(\Omega)^2$ and $H_0^1(\Omega)^2$.

Finite Element Subspaces

- $X_h \subset H_0^1(\Omega) \cap C(\bar{\Omega})$: approximation of the velocity u,

$$\inf_{\xi \in X_h} |v - \xi|_1 \leq C_0 h |v|_2 \quad \forall v \in H_0^1(\Omega) \cap H^2(\Omega).$$

- $S_h \equiv X_h^2$.
- $Y_h \subset L^2(\Omega) \cap C(\bar{\Omega})$: approximation of the pressure p.

• a priori error estimates for Stokes equations

$\forall f \in H^{-1}(\Omega)^2$, $[u, p] \in H_0^1(\Omega)^2 \times L_0^2(\Omega)$ satisfies

$$\nu(\nabla u, \nabla v) - (p, \operatorname{div} v) - (q, \operatorname{div} u) =< f, v > \quad \forall [v, q] \in H_0^1(\Omega)^2 \times L_0^2(\Omega).$$

and $[u_h, p_h] \in S_h \times Y_h$ satisfies

$$\nu(\nabla u_h, \nabla v_h) - (p_h, \operatorname{div} v_h) - (q_h, \operatorname{div} u_h) =< f, v_h > \quad \forall [v_h, q_h] \in S_h \times Y_h.$$

$\Longrightarrow$

$\exists C_1, C_2 > 0$ such that

$$\begin{aligned} |u - u_h|_1 &\leq C_1 |f|_0 \quad \text{if} \quad f \in L^2(\Omega)^2, \\ |u - u_h|_1 &\leq C_2 |f|_{-1} \quad \text{if} \quad f \in H^{-1}(\Omega)^2, \end{aligned}$$

where C_0, C_1 and C_2 are positive constant which can be numerically determined (proof is in JCAM **91** (1998) 137–158).

Finite Element Approximation

Finite element approximation $[u_h, p_h]$ for the Navier-Stokes equations:

$$\begin{cases} \nu(\nabla u_h, \nabla v_h) - (p_h, \operatorname{div} v_h) = -((u_h \cdot \nabla)u_h, v_h) + (f, v_h) & \forall v_h \in S_h, \\ -(\operatorname{div} u_h, q_h) = 0 & \forall q_h \in Y_h. \end{cases}$$

Here, $[u_h, p_h]$coincides with the finite element solution to the Stokes equation:

$$\begin{cases} -\Delta \bar{u} + \nabla \bar{p} = -(u_h \cdot \nabla)u_h + f & \text{in} \quad \Omega, \\ \operatorname{div} \bar{u} = 0 & \text{in} \quad \Omega, \\ \bar{u} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

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Residual Formulation

$$v_0 \equiv \bar{u} - u_h, \qquad p_0 \equiv \bar{p} - p_h,$$

$[v_0, p_0]$ corresponds to the residual error for the approximation $[u_h, p_h]$, then H_0^1 and L^2 norms for v_0 can be obtained in both of the a posteriori and a priori sense because of $-(u_h \cdot \nabla)u_h + f \in L^2(\Omega)^2$ (see *Reliable Computing* **4** (1998) 115–124).

v_0 is considered as an element in $H_0^1(\Omega)^2$ having the norm bounded, though its explicit form is unknown.

• Residual Form

$$\begin{cases} -\nu\Delta w + \nabla r = g(w) & \text{in} \quad \Omega, \\ \operatorname{div} w = 0 & \text{in} \quad \Omega, \\ w = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where

$$g(w) \equiv -((u_h + v_0 + w) \cdot \nabla)(u_h + v_0 + w) + (u_h \cdot \nabla)u_h.$$

Fixed Point Formulation

For each $\xi \in H^{-1}(\Omega)^2$, the Stokes problem

$$\begin{cases} -\nu\Delta\hat{w} + \nabla\hat{r} &= \xi \quad \text{in} \quad \Omega, \\ \operatorname{div}\hat{w} &= 0 \quad \text{in} \quad \Omega, \\ \hat{w} &= 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

has a unique solution $\hat{w} \in H_0^1(\Omega)^2$. For $\xi \in H^{-1}(\Omega)^2$, let $A\xi$ be the solution $\hat{w}$. Similarly, for $\xi \in H^{-1}(\Omega)^2$, there exists a unique finite element solution for the Stokes equation $\xi_h \in S_h$, and we define this correspondence by A_h.

- A is a continuous linear operator from $H^{-1}(\Omega)^2$ to $H_0^1(\Omega)^2$.
- A_h is a continuous linear operator from $H^{-1}(\Omega)^2$ to S_h.

Thus, setting compact map

$$F \equiv Ag,$$

we can rewrite the residual form as the fixed point problem in $H_0^1(\Omega)^2$:

$$w = Fw.$$

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Newton-like Method

• Product Space

$$S_h^* \equiv \{v \in H_0^1(\Omega)^2 \mid v = (A - A_h)f, \quad f \in H^{-1}(\Omega)^2\}$$

$X \equiv (S_h, S_h^*)$: product space,

$Px \equiv x_h + x^* \qquad x = (x_h, x^*) \in X,$

$G \equiv g \circ P.$

$\tilde{F}x \equiv (A_h Gx, (A - A_h)Gx) \quad : X \longrightarrow X$ compact map

• Schauder's Fixed Point Theorem

If we find a nonempty, bounded, convex and closed set $W \subset X$ such that $\tilde{F}W \subset W$, then there exists a fixed point x of $\tilde{F}$ in W. Moreover, for this fixed point $x = (x_h, x^*) \in X$, $Px = x_h + x^* \in H_0^1(\Omega)$ is also a fixed point of F.

• Newton-like Method

$$N_h x \equiv x_h - [P_1 - A_h \hat{g}'(u_h)]_h^{-1}(x_h - A_h Gx) \quad x = (x_h, x^*),$$

where P_1 is the H_0^1-projection from $H_0^1(\Omega)^2$ to S_h and $\hat{g}(w) \equiv -(w \cdot \nabla)w$.

$$Tx \equiv (N_h x, (A - A_h)Gx).$$

T is the compact operator on X and

$$x = Tx \Longleftrightarrow x = \tilde{F}x.$$

Computable Verification Condition

For any $v_h \in S_h = X_h^2$, using real coefficients $\{a_i\}_{1\le i\le 2n}$ and basis of X_h : $\{\phi_i\}_{1\le i\le n}$, we can write

$$v_h = (\sum_{i=1}^{n} a_i\phi_i, \sum_{i=1}^{n} a_{n+i}\phi_i)^T,$$

where $\dim X_h = n$.

$$(v_h)_i \equiv |a_i|.$$

• Candidate Set

$$W_h \equiv \{w_h \in S_h \mid (w_h)_i \le W_i \quad 1 \le i \le 2n\},$$

$$W^* \equiv \{w_1 + w_2 \in S_h^* \mid w_1, w_2 \in S_h^*,\ |w_1|_1 \le W_{2n+1}, |w_2|_1 \le W_{2n+2}\},$$

$$W \equiv (W_h, W^*) \subset X.$$

THEOREM(VERIFICATION CONDITION)

$$\begin{cases} N_h(W) & \subset & W_h \\ (A - A_h)G(W) & \subset & W^* \end{cases}$$

$\Longrightarrow$ there exists a fixed point x of $\tilde{F}$ in W.

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Computer Algorithm

$\underline{N = 0}$

$$W_i^{(0)} = 0 \quad 1 \le i \le 2n+2, \qquad W^{(0)} = (W_h^{(0)}, W^{*(0)}).$$

$\underline{N \ge 1}$

$$\bar{W}_i^{(N-1)} \equiv W_i^{(N-1)}(1+\delta) \qquad \bar{W}^{(N-1)} = (\bar{W}_h^{(N-1)}, \bar{W}^{*(N-1)}).$$

$$\begin{cases} W_h^{(N)} & \equiv & N_h \bar{W}^{(N-1)}, \\ W_{2n+1}^{(N)} & \equiv & C_1 \sup\limits_{w \in \bar{W}^{(N-1)}} |G_1(w)|_0, \\ W_{2n+2}^{(N)} & \equiv & C_2 \sup\limits_{w \in \bar{W}^{(N-1)}} |G_2(w)|_{-1}, \end{cases}$$

$$G_1(w) \equiv G(w) - G_2(w) \ \in L^2(\Omega)^2,$$

$$G_2(w) \equiv -((v_0 + w^*) \cdot \nabla)(v_0 + w^*) \ \in H^{-1}(\Omega)^2, \quad w = (w_h, w^*) \in X.$$

$$W^{(N)} = (W_h^{(N)}, W^{*(N)}).$$

Theorem(Verification Condition)

If, for a step K, $W_i^{(K)} < \bar{W}_i^{(K-1)} \quad 1 \le \forall i \le 2n+2$ hold, then in the set $\bar{W}^{(K-1)} = (\bar{W}_h^{(K-1)}, \bar{W}^{*(K-1)}) \subset X$ constructed by $\{\bar{W}_i^{(K-1)}\}_{1 \le i \le 2n+2}$, there exists an element x satisfying $x = Tx$.

Numerical Examples

$$\begin{cases} -\nu\Delta u + \nabla p &= -(u\cdot\nabla)u + f, \\ \operatorname{div} u &= 0, \end{cases}$$

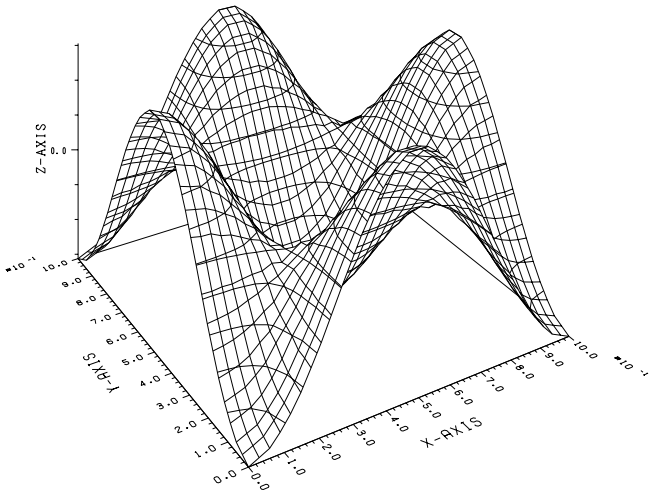
- $\Omega = (0,1)\times(0,1)$.
- uniform partition. L denotes the number of partitions for the interval $(0,1)$, i.e. $h = 1/L$.
- X_h : continuous piecewise quadratic polynomials, Y_h : piecewise linear.
- $C_0 = 1/(2\pi)$, $C_2 = 2/\nu$ and C_1 =a priori constant for Stokes equations.
- $f = (f_1, f_2)^T$ is given so that

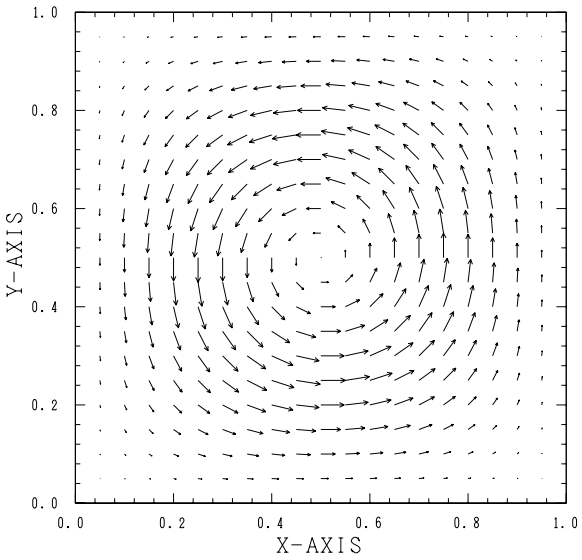
$$\begin{aligned} u(x,y) &= C(\sin^2\pi x \sin\pi y\cos\pi y, -\sin^2\pi y\sin\pi x\cos\pi x)^T, \\ p(x,y) &= -C^2\cos 2\pi x\cos 2\pi y/16 \end{aligned}$$

are the exact solutions.

$$|u|_1 = \frac{C\pi}{\sqrt{2}}, \quad |p|_0 = \frac{C^2}{32}.$$

Pressure and Vector Field





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Numerical Environment

- machine: FUJITSU VPP700/56
- software: Fortran 90/VP, Fortran 90/VPP(parallel compiler), *Mathematica*
- double precision (partly by quadruple precision)
  The round-off errors in these examples are neglected. Namely, the results are NOT validated in the sense of precise interval arithmetic.
- linear computation
  - linear equation(blocking LU decomposition)
  - matrix inversion(blocking LU decomposition)
  - generalized eigenvalue problem (Cholesky decomposition , Householder-bisection-inverse iteration method, Jacobi method)

Verified Examples

$u = u_h + v_0 + w$

Re	L	N	$\|u\|_1$	$\|v_0\|_1$	$\|\|W_h^{(N)}\|\|_\infty$	$W_{2n+1}^{(N)}$	$W_{2n+2}^{(N)}$
1	10	14	3.33×10^{-1}	2.61×10^{-2}	3.52×10^{-3}	2.44×10^{-3}	1.67×10^{-4}
2	10	12	2.22×10^{-1}	1.29×10^{-2}	1.33×10^{-3}	7.67×10^{-4}	7.67×10^{-5}
5	40	22	2.22×10^{-1}	3.55×10^{-3}	1.38×10^{-3}	7.59×10^{-4}	1.89×10^{-5}
10	40	23	1.11×10^{-1}	4.19×10^{-2}	4.26×10^{-4}	1.86×10^{-4}	1.47×10^{-6}
20	40	32	5.55×10^{-2}	5.81×10^{-4}	4.45×10^{-4}	1.96×10^{-4}	2.47×10^{-6}
40	40	27	2.22×10^{-2}	4.35×10^{-4}	2.90×10^{-4}	1.08×10^{-4}	2.42×10^{-6}
50	40	13	1.11×10^{-2}	2.68×10^{-4}	8.35×10^{-5}	1.95×10^{-5}	8.48×10^{-7}
100	40	15	4.44×10^{-3}	2.09×10^{-4}	6.63×10^{-5}	1.37×10^{-5}	1.02×10^{-6}

$Re = 1/\nu$.

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Verified Examples

Re	L	$\|\|u_h\|\|_\infty$	$\|f\|_0$	C_1	C_2
1	10	7.50×10^{-2}	2.56×10^{0}	1.40×10^{-1}	2
2	10	5.00×10^{-2}	3.41×10^{-1}	1.65×10^{-1}	4
5	20	5.00×10^{-2}	3.41×10^{-1}	1.43×10^{-1}	10
10	40	2.38×10^{-2}	8.54×10^{-2}	1.00×10^{0}	20
20	40	1.25×10^{-2}	2.13×10^{-2}	2.57×10^{-1}	40
40	40	5.00×10^{-3}	4.27×10^{-3}	5.11×10^{-1}	80
50	40	2.50×10^{-3}	1.70×10^{-3}	6.39×10^{-1}	100
100	40	1.00×10^{-3}	3.41×10^{-4}	1.27×10^{0}	200

$Re = 1/\nu$.

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