非線形楕円型境界値問題の解に対する精度保証付き計算法

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非線形楕円型方程式

$$\begin{cases} -\Delta u = \lambda f(u) & in \quad \Omega \\ u = 0 & on \quad \partial\Omega \end{cases} \tag{1}$$

Ω : convex polygonal in R^n $(n = 1, 2)$

$f(u) : H_0^1(\Omega) \longrightarrow L^2(\Omega) : conti.$

$U : bdd.\ in\ H_0^1(\Omega) \rightarrow f(U) : bdd.\ in\ L^2(\Omega)$

$\lambda \in R$

weak form

$$\begin{cases} find \quad u \in H_0^1(\Omega) \cap H^2(\Omega) \quad s.t. \\ (\nabla u, \nabla v) = \lambda(f(u), v) \\ \forall v \in H_0^1(\Omega) \end{cases} \tag{2}$$

Newton-Like method

G : 逆ラプラス作用素

$$I : H^2(\Omega) \hookrightarrow H^1(\Omega) : compact$$
$$F \equiv IG\lambda f : H_0^1(\Omega) \longrightarrow H_0^1(\Omega) : compact$$

$$(2) \Longleftrightarrow u = Fu$$

$$S_h \subset H_0^1(\Omega) : \text{有限要素空間} (0 < h < 1)$$

$P_h : H_0^1(\Omega)$からS_hへの直交射影

$$(\nabla(u - P_h u), \nabla v) = 0 \qquad \forall v \in S_h$$

u_h : S_hの固定要素

$F'(u_h)$: Fのu_hでの Fréchet 微分

$T : H_0^1(\Omega) \longrightarrow H_0^1(\Omega) : condensing$

$Tu \equiv \{I - (\ [\ I - F'(u_h)\ ]_h^{-1} P_h + \varepsilon I\ )(\ I - F\ )\}u$

$$u = Fu \Longleftrightarrow u = Tu$$

rounding

$\tilde{T}u \equiv \{\tilde{I} - ([\ I - F'(u_h)\ ]_h^{-1} + \varepsilon\tilde{I})(\tilde{I} - \tilde{F})\}u$

$\tilde{I} = P_h I\ ,\ \tilde{F} = P_h F\ ,\ u \in H_0^1(\Omega)$

$$R(TU) \equiv \{v \in S_h \mid v = \tilde{T}u\ ,\ u \in U\}$$

$$RE(TU) \equiv \left\{ \phi \in H_0^1(\Omega) \cap S_h^{\perp} \quad | \quad \|\phi\|_{H_0^1(\Omega)} \le \alpha \quad and \quad \|\phi\|_{L^2(\Omega)} \le Ch\alpha \right\}$$

$$where \qquad \alpha \equiv \sup_{u \in U} \|Tu - \tilde{T}u\|_{H_0^1(\Omega)}$$

Lemma 1

$U \subset H_0^1(\Omega)$: 有界凸閉集合

$$R(TU) \oplus RE(TU) \overset{\circ}{\subset} U$$
$$\Longrightarrow \exists u \in U \quad s.t. \quad u = Fu$$

計算機による検証手順

$$U = u_h + \Delta u_h \oplus [\alpha]$$

$$\Delta u_h = \sum_{j=1}^{M} A_j \phi_j \quad \in \Theta_h$$

$$A_j \equiv [\ \underline{A_j}\ ,\ \overline{A_j}\ ] \quad (1 \le j \le M)$$

$$[\alpha] \equiv \left\{ \phi \in H_0^1(\Omega) \cap S_h^{\perp} \quad | \quad \|\phi\|_{H_0^1(\Omega)} \le \alpha \quad and \quad \|\phi\|_{L^2(\Omega)} \le Ch\alpha \right\}$$

$\{\phi_j\}_{j=1,\cdots,M}$: S_h の基底

Θ_h : $\{\phi_j\}_{j=1,\cdots,M}$ と
区間係数との一次結合全体

反復法

$$U^{n-1} = u_h + \Delta u_h^{n-1} \oplus [\alpha_{n-1}]$$

$$\left\{ \begin{array}{l} \Delta \tilde{u}_h^{n-1} \equiv \Delta u_h^{n-1} + \sum_{j=1}^{M} [-1, 1]\, \delta\, \phi_j \\ \tilde{\alpha}_{n-1} \equiv \alpha_{n-1} + \delta \end{array} \right.$$

$$(\delta - inflation)$$

$$\tilde{U}^{n-1} \equiv u_h + \Delta \tilde{u}_h^{n-1} \oplus [\tilde{\alpha}_{n-1}]$$

検証完了条件

$$\left\{ \begin{array}{l} \Delta u_h^n \equiv \tilde{T}\tilde{U}^{n-1} - u_h \\ \alpha_n \equiv Ch\lambda \sup_{u \in \tilde{U}^{n-1}} \|f(u)\|_{L^2} \end{array} \right.$$

Theorem 1

$$U^n \equiv u_h + \Delta u_h^n \oplus [\alpha_n]$$

$$\left\{ \begin{array}{rcl} \Delta u_h^n & \overset{\circ}{\subset} & \Delta \tilde{u}_h^{n-1} \\ \alpha_n & < & \tilde{\alpha}_{n-1} \end{array} \right. \tag{3}$$

$$\Longrightarrow \ \exists u \in U^n \quad s.t. \quad u = Fu$$

検証法の実現

$$\Delta u_h^n = \sum_{j=1}^{M} (a_j + B_j^n)$$

$$\sum_{j=1}^{M} a_j \phi_j = -([I - F'(u_h)]_h^{-1} + \varepsilon \tilde{I})(\tilde{I} - \tilde{F}) u_h$$

$$\sum_{j=1}^{M} B_j^n \phi_j = \triangle \tilde{u}_h^{n-1} - ([I - F'(u_h)]_h^{-1} + \varepsilon \tilde{I})\{(\tilde{I} - \tilde{F})\tilde{U}^{n-1} - (\tilde{I} - \tilde{F}) u_h\}$$

$$\begin{aligned}
\Delta \tilde{u}_h^{n-1} &= \sum_{j=1}^{M} A_j^{n-1} \phi_j \\
u_h &= \sum_{j=1}^{M} b_j \phi_j \\
\tilde{F} u_h &= \sum_{j=1}^{M} c_j \phi_j \\
G &= (\nabla [I - F'(u_h)]_h \phi_j, \nabla \phi_i)_{j,i} \\
D &= (\nabla \phi_j, \nabla \phi_i)_{j,i} \\
K_i &= \lambda(k(u_h, \triangle \tilde{u}_h^{n-1}, [\tilde{\alpha}_{n-1}]), \phi_i)
\end{aligned}$$

$$k(u_h, \triangle \tilde{u}_h^{n-1}, [\tilde{\alpha}_{n-1}]) = f(\tilde{U}^{n-1}) - f(u_h) - f'(u_h) \triangle \tilde{u}_h^{n-1}$$

Proposition 1

$$\begin{aligned}
(a_j) &= (G^{-1} D + \varepsilon E)(c_i - b_i) \\
(B_i^n) &= -\varepsilon D^{-1} G (A_j) + (G^{-1} + \varepsilon D^{-1})(K_j)
\end{aligned}$$

数値例・１次元

$$\begin{cases} -u'' = \lambda u(u-a)(1-u) & in \ \ (0,1) \\ u(0) = 0 \ \ , \ \ u(1) = 0 \end{cases}$$

$$0 < a < \frac{1}{2}$$

$$x_i = \frac{i}{N} \quad (i = 0, 1 \ldots\ldots N)$$

$$\Omega_i = (x_{i-1}, x_i) \quad (i = 1, 2 \ldots\ldots N)$$

$$h = \frac{1}{N}$$

$P_1(\Omega_i)$ ： Ω_i上一次となる多項式の集合

$$S_h \equiv \{ \, v \in C(\Omega) \, ; \ v|_{\Omega_i} \in P_1(\Omega_i) \, , 1 \le i \le N \, , v(0) = v(1) = 0 \, \}$$

$$M \equiv \dim S_h = N - 1$$

$$\phi_j(x_k) = \begin{cases} 1 \ \ (k = j) \\ \\ 0 \ \ (k \neq j) \end{cases} \qquad for \ 1 \le j, k \le M \ .$$

$$C = \frac{1}{\pi}$$

数値例・2 次元

$$\begin{cases} -\Delta u &= \lambda u(u-a)(1-u) & \text{in} \quad \Omega \\ u &= 0 & \text{on} \quad \partial\Omega \end{cases}$$

$$\Omega = (0,1) \times (0,1)$$

$$\delta_x \ : \ 0 = x_0 < x_1 < \cdots x_n = 1$$

$$\delta \ : \ \delta = \delta_x \otimes \delta_y$$

$$S_h \equiv \mathcal{M}_0^1(x) \otimes \mathcal{M}_0^1(y)$$

$$M \equiv \dim S_h = (N-1)^2$$

$$C = \frac{1}{\pi}$$

$$\delta = 10^{-6}$$

$$\varepsilon = 10^{-4}$$

$$\triangle u_h^0 = 0$$

$$\alpha_0 = 0$$

数値例・2 次元

$$\begin{cases} -\Delta u &= u^2 & \text{in} \quad \Omega \\ u &= 0 & \text{on} \quad \partial\Omega \end{cases}$$

$$\Omega = (0,1) \times (0,1)$$

$u_h \in S_h$ ：近似解

$\omega \in H^2(\Omega)$ ： u_h の Helmite 補間

$$\begin{cases} -\Delta v &= v^2 + 2\omega v + \Delta\omega + \omega^2 & \text{in} \quad \Omega \\ v &= 0 & \text{on} \quad \partial\Omega \end{cases}$$

検証例・2 次元

$f(u) = u^2$

分割数　$N = 80$

S_h の次元 $M = 6241$

反復数 12

$\alpha = 0.06661340$