Verified Computation of Solutions for Nondifferentiable Elliptic Equations Related to MHD Equilibria

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Semilinear elliptic problem

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{1}$$

Ω : convex polygonal in R^2

$f(u) : H_0^1(\Omega) \longrightarrow L^2(\Omega)$: conti.

U : bdd. in $H_0^1(\Omega) \rightarrow f(U)$: bdd. in $L^2(\Omega)$

$\lambda \in R$

$f(t) = (t-1)^+ \equiv \max\{t, 0\}$

weak form

$$\begin{cases} \text{find} \quad u \in H_0^1(\Omega) \cap H^2(\Omega) \quad \text{s.t.} \\ \qquad (\nabla u, \nabla v) = \lambda(f(u), v), \\ \qquad\qquad \forall v \in H_0^1(\Omega). \end{cases} \tag{2}$$

Newton-like method

$$G \in \mathcal{L}(L^2(\Omega); H_0^1(\Omega) \cap H^2(\Omega))$$

for $f \in L^2(\Omega)$, $Gf \equiv v$ is the solution of $-\Delta v = f$.

$$I : H^2(\Omega) \hookrightarrow H^1(\Omega) : \text{compact}$$

$$F \equiv IG\lambda f : H_0^1(\Omega) \longrightarrow H_0^1(\Omega) : \text{compact}$$

$$(2) \Longleftrightarrow u = Fu$$

$S_h \subset H_0^1(\Omega)$: finite element space $(0 < h < 1)$

P_h : orthogonal projection from $H_0^1(\Omega)$ to S_h.

$$(\nabla(u - P_h u), \nabla v) = 0, \qquad \forall v \in S_h$$

$u_h \in S_h$

$F'(u_h)$: Fréchet-like derivative on u_h

$$(\nabla F'(u_h)u, \nabla w) = (f'(u_h)u, w)$$

$$f'(t) \equiv \begin{cases} 1 & t > 1 \\ \xi & t = 1 \\ 0 & t < 1 \end{cases}$$

$$T_\varepsilon : H_0^1(\Omega) \longrightarrow H_0^1(\Omega) : \text{condensing}$$

$$T_\varepsilon u \equiv \{I - (\ [\ I - F'(u_h)\ ]_h^{-1} P_h + \varepsilon I\ )(\ I - F\ )\}u$$

$$u = Fu \Longleftrightarrow u = T_\varepsilon u$$

rounding

$$\tilde{T}_\varepsilon u \equiv \{\tilde{I} - ([\ I - F'(u_h)\ ]_h^{-1} + \varepsilon\tilde{I})(\tilde{I} - \tilde{F})\}u$$
$$\tilde{I} = P_h I\ ,\ \tilde{F} = P_h F\ ,\ u \in H_0^1(\Omega)$$

$$R\big(T_\varepsilon U\big) \equiv \{v \in S_h \mid v = \tilde{T}_\varepsilon u\ ,\ u \in U\}$$

$$RE\big(T_\varepsilon U\big) \equiv \left\{ \begin{array}{cc} \phi \in H_0^1(\Omega) \cap S_h^\perp & \mid \quad \|\phi\|_{H_0^1(\Omega)} \le \alpha \\ & and\ \ \|\phi\|_{L^2(\Omega)} \le Ch\alpha \end{array} \right\}$$

$$where \quad \alpha \equiv \sup_{u \in U} \|T_\varepsilon u - \tilde{T}_\varepsilon u\|_{H_0^1(\Omega)}$$

Lemma 1

$U \subset H_0^1(\Omega)$: bounded convex closed set

$$R\big(T_\varepsilon U\big) \oplus RE\big(T_\varepsilon U\big) \overset{\circ}{\subset} U$$

$$\Longrightarrow\ \exists u \in U \quad s.t. \quad u = Fu.$$

verification condition

$$U = u_h + \delta u_h \oplus [\alpha]$$

$$\delta u_h = \sum_{j=1}^{M} A_j \phi_j \quad \in \Theta_h$$

$$A_j \equiv [\ \underline{A_j}\ ,\ \overline{A_j}\ ] \quad (1 \le j \le M)$$

$$[\alpha] \equiv \left\{ \begin{array}{ll} \phi \in H_0^1(\Omega) \cap S_h^{\perp} & | \quad \|\phi\|_{H_0^1(\Omega)} \le \alpha \\ & and \ \ \|\phi\|_{L^2(\Omega)} \le Ch\alpha \end{array} \right\}$$

$\{\phi_j\}_{j=1,\cdots,M}$: basis of S_h

Θ_h : set of all linear combinations of $\{\phi_j\}_{j=1,\cdots,M}$ with interval coefficients

iteration method

$$U^{n-1} = u_h + \delta u_h^{n-1} \oplus [\alpha_{n-1}]$$

$$\left\{ \begin{array}{l} \delta \tilde{u}_h^{n-1} \equiv \delta u_h^{n-1} + \sum_{j=1}^{M} [-1, 1]\, \delta\, \phi_j \\ \tilde{\alpha}_{n-1} \equiv \alpha_{n-1} + \delta \end{array} \right.$$

$$\left(\delta - inflation\right)$$

$$\tilde{U}^{n-1} \equiv u_h + \delta \tilde{u}_h^{n-1} \oplus [\tilde{\alpha}_{n-1}]$$

verification lemma

$$\begin{cases} \delta u_h^n \equiv \tilde{T}_\varepsilon \tilde{U}^{n-1} - u_h \\ \alpha_n \equiv Ch\lambda \sup\limits_{u \in \tilde{U}^{n-1}} \|f(u)\|_{L^2} \end{cases}$$

Theorem 1

$U^n \equiv u_h + \delta u_h^n \oplus [\alpha_n]$

$$\begin{cases} \delta u_h^n \overset{\circ}{\subset} \delta \tilde{u}_h^{n-1} \\ \alpha_n < \tilde{\alpha}_{n-1} \end{cases} \tag{3}$$

$\Longrightarrow \quad \exists u \in U^n \quad s.t. \quad u = Fu$

Algorithm

$$\delta u_h^n = \sum_{j=1}^{M} \left(a_j + B_j^n \right)$$

$$\sum_{j=1}^{M} a_j \phi_j = -([I - F'(u_h)]_h^{-1} + \varepsilon \tilde{I})(\tilde{I} - \tilde{F}) u_h$$

$$\sum_{j=1}^{M} B_j^n \, \phi_j = \delta \tilde{u}_h^{n-1} - ([I - F'(u_h)]_h^{-1} + \varepsilon \tilde{I})\{(\tilde{I} - \tilde{F})\tilde{U}^{n-1} - (\tilde{I} - \tilde{F}) u_h\}$$

$$\begin{aligned}
\delta \tilde{u}_h^{n-1} &= \sum_{j=1}^{M} A_j^{n-1} \phi_j \\
u_h &= \sum_{j=1}^{M} b_j \phi_j \\
\tilde{F} u_h &= \sum_{j=1}^{M} c_j \phi_j \\
G &= \left(\nabla \left[I - F'(u_h) \right]_h \phi_j, \nabla \phi_i \right)_{j,i} \\
D &= \left(\nabla \phi_j, \nabla \phi_i \right)_{j,i} \\
K_i &= \lambda \left(k\left(u_h, \delta \tilde{u}_h^{n-1}, [\tilde{\alpha}_{n-1}] \right), \phi_i \right)
\end{aligned}$$

$$k(u_h, \delta \tilde{u}_h^{n-1}, [\tilde{\alpha}_{n-1}]) = f(\tilde{U}^{n-1}) - f(u_h) - f'(u_h) \delta \tilde{u}_h^{n-1}$$

Proposition 1

$$
\begin{aligned}
(a_j) &= (G^{-1}D + \varepsilon E)(c_i - b_i) \\
(B_i^n) &= -\varepsilon D^{-1} G(A_j) + (G^{-1} + \varepsilon D^{-1})(K_j)
\end{aligned}
$$

MHD equilibria

$$\begin{cases} -\Delta u = \lambda u^{+} & \text{in } \Omega \\ u = -1 & \text{on } \partial\Omega \end{cases}$$

$$\Omega = (0,1) \times (0,1)$$

$$u^{+}(x) = \max\{0, u(x)\}$$

$$\begin{cases} -\Delta w = \lambda (w-1)^{+} & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases}$$

$$\delta_x \; : \; 0 = x_0 < x_1 < \cdots \cdot x_n = 1$$

$$\delta \; : \; \delta = \delta_x \otimes \delta_y$$

$$S_h \equiv \mathcal{M}_0^1(x) \otimes \mathcal{M}_0^1(y)$$

$$M \equiv \dim S_h = (N-1)^2$$

$$C = \frac{1}{\pi}$$

$$\triangle u_h^0 = 0$$

$$\alpha_0 = 0$$

計算手順

$$f(t) = \begin{cases} t-1 & t > 1 \\ 0 & t \leq 1 \end{cases}$$

$$f'(t) \equiv \begin{cases} 1 & t > 1 \\ \xi & t = 1 \\ 0 & t < 1 \end{cases}$$

K_i の計算

$$(\nabla F'(u_h)\phi_k, \nabla\phi_j) = (f'(u_h)\phi_k, \phi_j)$$

$$\left((u_h+\delta u_h^{n-1}-1)^+ - (u_h-1)^+ - f'(u_h)\delta u_h^{n-1}, \phi_i \right)_{\Omega_k}$$

$$d \equiv ||\delta u_h^{n-1}||_{L^\infty(\Omega_k)}$$

積分領域　$\{x \in \Omega_k \ : \ -d \leq u_h - 1 \leq d\}$

Example

$\lambda = 34.0898$
number of partitions $N = 120$
$\dim S_h = 14161$
iteration number 10
L^2 error $= 0.0002919704$

Example

$\lambda = 27.00$
number of partitions $N = 130$
$\dim S_h = 16641$
iteration number 7
L^2 error$= 0.0001910316$

$\{\phi_j\}_{j=1,\cdots,M}$: basis of S_h

Θ_h : set of all linear combinations of $\{\phi_j\}_{j=1,\cdots,M}$ with interval coefficients

NOTE

each element of Θ_h is defined as a subset of S_h, namely, $u \in \Theta_h$ means that

$$\omega = \sum_{j=1}^{M} A_j \phi_j \equiv \{ \sum_{j=1}^{M} a_j \phi_j \mid a_j \in A_j \}$$

where A_j are real intervals.

For any $\psi \in L^2(\Omega)$, the boundary value problem:

$$\begin{cases} -\Delta\phi = \psi, & \text{in } \Omega. \\ \phi = 0, & \text{on } \partial\Omega. \end{cases} \tag{4}$$

has a unique solution $\phi \in H_0^1(\Omega) \cap H^2(\Omega)$ and that the following estimates hold:

$$| \phi |_{H^2(\Omega)} \leq C_1 \| \psi \|_{L^2(\Omega)}.$$

where C_1 is a positive constant independent of ψ, and $|\cdot|_{H^2(\Omega)}$ implies the semi-norm.