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Verified Computation of Solutions for Nondifferentiable Elliptic Equations Related to MHD Equilibria

We propose a numerical technique to enclose the exact solution with guaranteed error bounds for nonlinear elliptic equations of second order with nondifferentiable term related to MHD equilibria. Our method is a kind of Newton-like method using an enclosing technique combined with the explicit error estimate for finite element approximations. We construct a computing algorithm which automatically verify the solution with guaranteed error bounds.

1. Problem and fixed point formulation

We consider the following nonlinear elliptic problem:

$$\begin{cases} -\Delta u &= \lambda f(u) & \text{in} \quad \Omega, \\ u &= 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1}$$

where Δ is the Laplace operator, and Ω is a bounded and convex domain in $\mathbb{R}^n (1 \le n \le 3)$ with piecewise smooth boundary $\partial\Omega$, $\lambda \ge 0$ a real parameter and $f(t) \equiv (t-1)^+ = \max\{0, t-1\}$. (1) arises in equilibrium analysis of confined MHD (magnetohydrodynamics) plasmas or thin stretched membranes partially covered with water. Clearly, the function $u(x) \equiv 0$, $x \in \Omega$ satisfies (1) for all λ, and we will seek other non trivial solutions. Let $H^m(\Omega)$ denote m-th order L^2-Sobolev space on Ω, and set $H_0^1(\Omega) = \{ v \in H^1(\Omega) \mid v = 0 \text{ on } \partial\Omega \}$. The inner product on $H_0^1(\Omega)$ is defined as $(\nabla u, \nabla v)$, where $(\cdot, \cdot)$ is the $L^2(\Omega)$-inner product. We can show the nonlinear real-valued function $f(\cdot)$ is the continuous bounded map from $H_0^1(\Omega)$ to $L^2(\Omega)$. Next, we rewrite (1) in weak form:

$$\text{find} \quad u \in H_0^1(\Omega) \cap H^2(\Omega) \quad \text{s.t.} \quad (\nabla u, \nabla v) = \lambda(f(u), v), \quad \forall v \in H_0^1(\Omega). \tag{2}$$

It is well known that for any $\psi \in L^2(\Omega)$, the boundary value problem:

$$-\Delta\phi = \psi \quad \text{in } \Omega, \qquad \phi = 0 \quad \text{on } \partial\Omega \tag{3}$$

has a unique solution $\phi \in H_0^1(\Omega) \cap H^2(\Omega)$ and the estimate $|\phi|_{H^2(\Omega)} \le C_1 \|\psi\|_{L^2(\Omega)}$ holds. Here C_1 is a positive constant independent of ψ, and $|\cdot|_{H^2(\Omega)}$ implies the $H^2(\Omega)$-seminorm . For $\psi \in L^2(\Omega)$, let $G\psi$ be the solution of (3). Then G is a compact operator from $H_0^1(\Omega)$ to $L^2(\Omega)$ beacuse of the compactness of the imbedding $H^2(\Omega) \hookrightarrow H^1(\Omega)$. Then the nonlinear operator $F \equiv G\lambda f$ is a compact operator on $H_0^1(\Omega)$, and the weak solution of (2) can be rewritten as the fixed point form $u = Fu$. Now, we will introduce a Newton-like method. Let us introduce an generalized derivative f' on $\mathbb{R}$ by

$$f'(t) \equiv \begin{cases} 1 & t > 1, \\ \xi & t = 1, \\ 0 & t < 1. \end{cases}$$

where ξ is a given fixed finite real value. Since $f'(u)w \in L^2(\Omega)$ for all $u, w \in H^1(\Omega)$, we can define the linear operator $F'(u)$ on $H_0^1(\Omega)$ for each $u \in H_0^1(\Omega)$ $F'(u)$ satisfing $(\nabla F'(u)w, \nabla v) = \lambda(f'(u)w, v), \quad \forall w, v \in H_0^1$. Next, let S_h be an appropriate finite element subspace of $H_0^1(\Omega)$ dependent on a parameter h ($0 < h < 1$) and $u_h \in S_h$ be an approximate solution to (2). Further, let P_h be an orthogonal projection from $H_0^1(\Omega)$ into S_h in $H_0^1(\Omega)$-sense. For a small parameter ε $(0 < \varepsilon < 1)$, we define the nonlinear operator T_ε on $H_0^1(\Omega)$ by

$$T_\varepsilon u \equiv \{ I - ([I - F'(u_h)]_h^{-1} P_h + \varepsilon I)(I - F) \} u. \tag{4}$$

Note that if the operator $[I - F'(u_h)]_h^{-1} P_h + \varepsilon I$ is invertible, the two fixed point form, $u = Fu$ and $u = T_\varepsilon u$ are equivalent. We can easily show that the operator T_ε is a condensing operator. Therefore, if there exist a non-empty, bounded convex and closed set U in $H_0^1(\Omega)$ satisfing $T_\varepsilon U \subset U$, then there exists a fixed point u of T in U by Sadovskii's fixed point theorem.

2. Computable verification condition

Let P_h be the H_0^1-projection and $\tilde{T}_\varepsilon \equiv P_h T_\varepsilon$. For a set $U \in H_0^1(\Omega)$, we define the rounding $R(T_\varepsilon U)$ as

$$R(T_\varepsilon U) \equiv \{v \in S_h \mid v = \tilde{T}_\varepsilon u \ , \ u \in U\}.$$

Next, we assume, as the approximation property of P_h, that

$$\|u - P_h u\|_{H_0^1(\Omega)} \le C_2 h |u|_{H^2(\Omega)}, \qquad \forall u \in H_0^1(\Omega) \cap H^2(\Omega),$$

where C_2 is a positive constant independent of u and h which can be numerically determined. Then, for $\alpha \equiv \sup_{u \in U} \|T_\varepsilon u - \tilde{T}_\varepsilon u\|_{H_0^1(\Omega)}$ and $C \equiv C_1 C_2$, we define the rounding error $RE(T_\varepsilon U)$ as

$$RE(T_\varepsilon U) \equiv \{\phi \in S_h^\perp \mid \|\phi\|_{H_0^1(\Omega)} \le \alpha \ \text{ and } \ \|\phi\|_{L^2(\Omega)} \le Ch\alpha\},$$

Now, we have the following computable verification condition.

Theorem: *Let $U \subset H_0^1(\Omega)$ be a non-empty bounded convex and closed subset such that for some $\varepsilon (0 < \varepsilon < 1)$, $R(T_\varepsilon U) \oplus RE(T_\varepsilon U) \overset{\circ}{\subset} U$, then there exists a solution of $u = Fu$ in U. Here, $\oplus$ denotes the direct sum in the sense of $H_0^1(\Omega)$ and $M_1 \overset{\circ}{\subset} M_2$ implies $\overline{M_1} \subset \overset{\circ}{M_2}$ for any sets M_1, M_2.*

In the actual verification, we use an iterative method for the operator T_ε, see [1] for details.

3. Numerical Example

Let Ω be a rectangle domain in $\mathbb{R}^2$ such that $\Omega = (0,1) \times (0,1)$. Also let $\delta_x : 0 = x_0 < x_1 < \cdots < x_N = 1$ be a uniform partition, and let δ_y be the same partition as δ_x for y direction. We define the partition of Ω by $\delta \equiv \delta_x \otimes \delta_y$. Further, we define the finite element subspace S_h by $S_h \equiv \mathcal{M}_0^1(x) \otimes \mathcal{M}_0^1(y)$, where $\mathcal{M}_0^1(x)$, $\mathcal{M}_0^1(y)$ are sets of continuous piecewise linear polynomials on $(0,1)$ in the variables x and y, respectively.

Example: $\lambda = 40$, number of partitions $N = 130$, iteration numbers= 5, H_0^1-error bound : $\alpha_5 = 0.02726$. Figure 1 shows the outline of the shape for the solution along line $y = 0.5$. That is, there exists a solution between those two curves with additional $H_0^1(\Omega)$-error α_5.

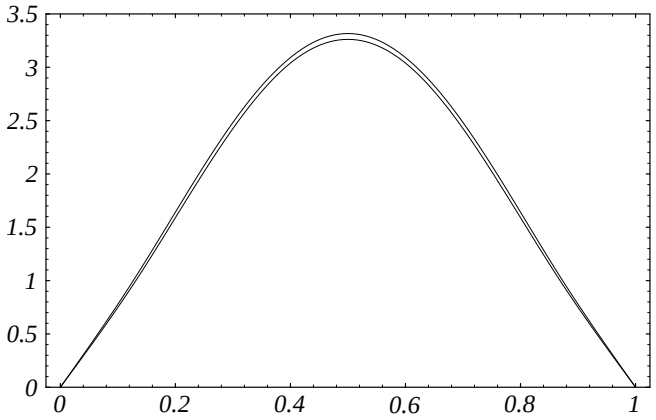

Figure 1. Outline of the shape for the solution along the line $y = 0.5$

4. References

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