Verified numerical computations for an inverse elliptic eigenvalue problem with finite data

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Abstract

We consider a numerical enclosure method for solutions of an inverse Dirichlet eigenvalue problem. When the finite number of prescribed eigenvalues are given, we recontruct a potential function, with guaranteed error bounds, for which the corresponding elliptic operator exactly has those eigenvalues including the ordering property. All computations are executed with numerical verifications based upon the finite and infinite fixed point theorems using interval arithmetic. Therefore, the results obtained are mathematically correct. We present a numerical example which confirms us the enclosure algorithm works on a real problem.

Keywords: Inverse elliptic eigenvalue problem, Numerical verification method, Computer assisted proof.

1 Introduction

The inverse eigenvalue problems are known as the problem to find a potential function so that the associated eigenvalue problem has the prescribed spectrum. Particularly, Neher [10],[11] studied a numerical enclosure method of solutions for the inverse Sturm-Liouville problem with finite data. Under the assumption that the potential is represented as a linear combination of some finite number of base functions, he presented an algorithm determining the coefficients of the linear combiation with guaranteed accuracy using the interval Newton method [1]. In the inverse eigenvalue problems, in general, the enclosing method of the exact eigenvalues and eigenfunctions for ordinary eigenvalue problems plays an essential role. Neher's technique utilizes an enclosure method for solutions of intial value problem for ordinary differential equations developed by Lohner [3] with shooting arguments as well as some theoretical results on the Sturm-Liouville problems. However, these techniques, except for the formulation of the problem, could not be applied to the multi-dimensional case, i.e., elliptic problems. This is the principal motivation of our work in this paper. In other words, our main result of this paper is an extension of Neher's work to the inverse elliptic eigenvalue problems with fimnite data by using quite different technique from [11].

We present a computing algorithm to reconstruct the potential function with verification from the finite number of prescribed eigenvalues as well as describe some theoretical results for the guaranteed computations. The eigenvalue and eigenfunction enclosure with preserving the ordering property of eigenvalues for ordinary problem would be the main task of the present work. In order to attain the purpose we used our numerical verification method for solutions of nonlinear elliptic problems developed by the authors' research for years. The authors believe that this result is the first approach in the world to the verified computation of solutions for the inverse elliptic eigenvalue problems.

2 Formulation of problem

Consider the following Dirichlet eigenvalue poblem :

$$\begin{cases} -\Delta u + qu &= \lambda u, \quad x \in \Omega, \\ u &= 0, \quad x \in \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded convex domain in R^2 and $q \in L^\infty(\Omega)$. Here, q is a C^0 function on $\bar{\Omega}$.

Let $\{\mu_i\}_{i=1,\cdots,M}$ be given finite data, real numbers, satisfying

$$\mu_1 < \mu_2 < \quad \cdots \quad < \mu_M.$$

Our aim is the finding a potential q in some set of functions so that the lowest i-th eigenvalue of (1) coinsides with μ_i for each i, $1 \le i \le M$.

We now define the set of functions

$$S \equiv \{q \in C^0(\bar{\Omega}) \mid q \equiv q(\alpha) := \hat{q} + \sum_{j=1}^{M} \alpha_j \phi_j, \ where \ \alpha = (\alpha_j)_{j=1,\cdots,M} \in R^M\}. \tag{2}$$

Here, $\{\phi_j\}_{j=1,\cdots,M}$ is a set of linearly independent functios in $C^0(\bar{\Omega})$ and $\hat{q}$ stands for a fixed function, approximate potential. For a vector $\alpha \equiv (\alpha_j)_{j=1,\cdots,M}$, we denote the corresponding i-th eigenvalue and eigenfunction for the problem (1) with $q = q(\alpha)$ by $\lambda_i(\alpha)$ and $u_i(\alpha)$, respectively.
Then, our problem can be written as follows: find $\exists \alpha \in R^M$ such that

$$f(\alpha) = 0, \tag{3}$$

where $f(\alpha) \equiv (f_1(\alpha), \cdots, f_M(\alpha))^T$ and each component is defined by

$$f_i(\alpha) \equiv \lambda_i(\alpha) - \mu_i.$$

Therefore, our final purpose is the enclosure of solutions for the M-dimensional nonlinear equation (3). However, notice that $\lambda_i(\alpha)$ is implicitly determined through the solutions of the following elliptic eigenvalue problem with $q = q(\alpha)$, for $1 \le i \le M$:

$$\begin{cases} -\Delta u_i + q u_i &= \lambda_i u_i, \quad x \in \Omega, \\ \\ u_i &= 0, \quad\quad x \in \partial\Omega. \end{cases} \tag{4}$$

Thus, in order for solving (3), we have to compute explicitly, or with guaranteed error bounds, the solutions of (4) for the given $\alpha \in R^M$.

We now introduce some notations for later use. Let denote usual m-th order L^2-Sobolev space on Ω by $H^m \equiv H^m(\Omega)$ and the first order space with homogeneous boundary condition by $H_0^1 \equiv H_0^1(\Omega)$. Further, define the space for the eigenpairs by $V := H_0^1 \times R$ with the canonical norm of product space. We denote the set of real intervals and n dimensional interval vectors by IR and IR^n, respectively.

3 Verification algorithm by the interval Newton's method

In order to attain our purpose, we need some interval functions for the potential q. Namely, for any $[\alpha] \equiv ([\alpha_i]) \in IR^M$, where $[\alpha_i] = [\underline{\alpha_i}, \overline{\alpha_i}]$, interval function $q([\alpha])$ of potentials is interpreted as:

$$\begin{aligned} q([\alpha]) &\equiv \hat{q} + \sum_{j=1}^{M} \alpha_j \phi_j \\ &:= \{\phi \in S \mid \phi = \hat{q} + \sum_{j=1}^{M} a_j \phi_j, \ \forall a_j \in [\alpha_j]\}. \end{aligned}$$

The mid point verctor of $[\alpha] \in IR^M$ is denoted by $m([\alpha])$. And, $\lambda_i([\alpha])$ and $u_i([\alpha])$ imply the set of i-th eigenvalues and eigenfunctions of (4) for all $\alpha \in [\alpha]$.
In order to enclose the solutions for (3), we use the interval Newton's method, similar to that in [11], which is described as below.

INTERVAL NEWTON'S METHOD FOR NONLINEAR EQUATION (3)

1. Choose $\hat{q}$, $[\alpha]^{(0)} \in IR^M$.
   $k := 0$

2. Setting $m^{(k)} \equiv m([\alpha]^{(k)})$, compute the followings with guaranteed accuracy, for $1 \le i \le M$:
   $\lambda_i(m^{(k)})$: eigenvalue of (4) with $q = q(m^{(k)})$,

   $u_i([\alpha])$: eigenfunction of (4) with $q = q([\alpha]^{(k)})$,

   and set
   $$f_i(m^{(k)}) := \lambda_i(m^{(k)}) - \mu_i.$$

3. Compute the Jacobian interval matrix:
   $$J := \left(\frac{\partial f_i}{\partial \alpha_j}([\alpha]^{(k)}) \right)$$

4. Execution of the interval Newton iteration:
   $$IN([\alpha]^{(k)}) := m^{(k)} - J^{-1} f(m^{(k)}).$$

5. If $IN([\alpha]^{(k)}) \subset [\alpha]^{(k)}$, then there exists exactly one $\alpha^* \in [\alpha]^{(k)}$ such that $f(\alpha^*) = 0$,
   otherwise, after setting $[\alpha]^{(k+1)} := IN([\alpha]^{(k)}) \cap [\alpha]^{(k)}$ and $k := k+1$, go to Step 2, when $k \le k_0$.

The normal completion of the above algorithm implies that the interval function $q([\alpha]^{(k)})$ contains a locally unique potential function of the form

$$q(\alpha^*) \equiv \hat{q} + \sum_{j=1}^{M} \alpha_j^* \phi_j$$

such that the eigenvalue problem

$$\begin{cases} -\Delta u + q(\alpha^*)u &= \lambda u, \quad x \in \Omega, \\ u &= 0, \quad x \in \partial\Omega, \end{cases} \tag{5}$$

exactly has the lowest M eigenvalues $\{\mu_i\}_{i=1,\cdots,M}$.

In the following sections, we describe the verified computational method for each quantity in the above algorithm. That is, the eigenvalue enclosing/excluding and the eigenfunction enclosing in Step 2 will be mentioned in Section 4 and 5, while the method of the verified computations for Jacobians in Step 3 will be discussed in Section 6 and 7.

Remark 3.1. We might take the following alternative procedure in Step 5 instead that $[\alpha]^{(k+1)} := IN([\alpha]^{(k)}) \cap [\alpha]^{(k)}$:

$$[\alpha]^{(k+1)} := [1-\varepsilon, 1+\varepsilon]IN([\alpha]^{(k)}) \quad (\varepsilon\text{-inflation, see, e.g., [12]}).$$

4 Eigenpair enclosing method

To execute the iterative process in the previous section, we numerically and rigorously enclose the eigenvalues and corresponding eigenfunctions for the following normalized eigenequation with a given potential function $q = q(\alpha)$.

$$\begin{cases} -\Delta u + qu &= \lambda u, \\ \int_\Omega u^2 &= 1. \end{cases} \tag{6}$$

In order to enclose the eigenpair (u, λ) for (6), we use the numerical verification method for solutions of elliptic problems developed in [6], [7], [8]etc. In our enclosure method, the equation (6) is considered as a nonlinear problem in (u, λ), and the eigenvalue and eigenfunction are simultaneously enclosed near the approximate eigenpair. Particularly, we use the enclosure method with uniqueness property which means that the assured interval exactly contains one eigenvalue, see [5] for details.

Furthermore, in the present case, for the execution of the Newton-iteration, we need to treat the interval valued function $q([\alpha])$ as described before. But this requirement causes no essential difficulty at all, for our method is originally based on a kind of interval method with additional considerations on the error from the gap between finite and infinite dimension. Thus we can enclose all eigenpairs around some neighbourhood of an approximate solution of the discretized problem for (6) with set of potential functions $q = q([\alpha])$ instead of $q(\alpha)$.

Also notice that our enclosure method in general presents the eigenvalue including interval I_{inc} and the uniqueness interval I_{uni} so that $I_{inc} \overset{\circ}{\subset} I_{uni}$, where $I_{inc} \overset{\circ}{\subset} I_{uni}$ implies that the closure of I_{inc} is covered by the interior of I_{uni}. Note that I_{inc} contains an unique eigenvalue and that there is no eigenvalue in the set $I_{uni} - I_{inc}$.

5 Eigenvalue excluding method

It is necessary to guarantee the order of M eigenvalues for (1) to get the actually lowest M eigenvalues. This can be done by the eigenvalue excluding technique which was studied in [9],[4], and is also closely related to the arguements in the section 7. We briefly remark here on the basic principle of the method.

For, usually rather narrow, an interval $\Lambda \in IR$ and, for an $\lambda \in \Lambda$, set

$$L(\lambda) \equiv -\Delta u + (q - \lambda)u.$$

Then, since $L(\lambda)$ is a linear elliptic operator, the following equation has a trivial solution $u = 0$:

$$\left\{ \begin{array}{rcll} L(\lambda)u &=& 0, & x \in \Omega, \\ u &=& 0, & x \in \partial\Omega. \end{array} \right. \tag{7}$$

Therefore, if we validate the uniqueness of the solution in (7), it implies that λ is not an eigenvalue of (1). That is, there is no eigenvalue of (7) in Λ. The method to prove this uniqueness is analogous to that in the section 7. Thus the eigenvalue excluding process advances from the one to the next, backward or forward to the adjacent interval. Since, by some eigenvalue shift, we can easily present the lower bound of the spectrum of (1), by an appropriately combining this excluding procedure with the enclosing technique described before, we obtain the eigenvalue ordering as far as each eigenvalue is geometrically simple.

We now note that, although we treat the interval potential function, it is sufficient to complete this eigenvalue excluding process only for the midpoint potenial $q(m[\alpha])$. That is, we can prove the following theorem.

Theorem 1. Assume that M eigenvalues are enclosed in the intervals $I_0^{(j)}$ with uniqueness in the disjoint M intervals $I^{(j)}$ $(1 \leq j \leq M)$ for the interval potential function $q([\alpha])$ (Fig.1). Further assume that the eigenvalue excluding process is completed for the midpoint potenial $q(m[\alpha])$. Then each $I^{(j)}$ exactly contains j-th eigenvalue for all $q(\alpha)$, $\alpha \in [\alpha]$. That is, the lowest M eigenvalues are validated for all potentials.

Proof. Let J_1 be the left most excluded interval for $q = q(m[\alpha])$ as in Fig.1. The real constant β in Fig.1 is chosen so that all elliptic operators $L_\beta \equiv -\Delta + (q([\alpha]) - \beta)$ are positive definite.
Now, if there exists an $\tilde{\alpha} \in [\alpha]$ such that $\lambda_1(\tilde{\alpha})$ is in the excluded interval J_1, then, $\lambda_2(\tilde{\alpha})$, in general $\lambda_j(\tilde{\alpha})$ for $2 \leq j$, has to be in the interval $I_0^{(1)}$ by the existential assumption on $I_0^{(1)}$. On the other hand, $\lambda_2(m[\alpha])$ is in the another interval $I_0^{(2)}$ apart from $I^{(1)}$(Fig.1). It implies that, by the continuity argument with parameter α, the second eigenvalues $\lambda_2(\alpha)$ has to exist continuously in the interval $[\lambda_2(\tilde{\alpha}), \lambda_2(m[\alpha])]$. But this interval covers I_{01}' in Fig.1, where no eigenvalue exists, which is a contradiction. And similar arguement can also be applied to the j-th eigenvalue for $2 \leq j \leq M-1$. Since it is not necessary any excluding argument for $j = M$ due to the uniqueness assumption, we conclude the proof. $\square$

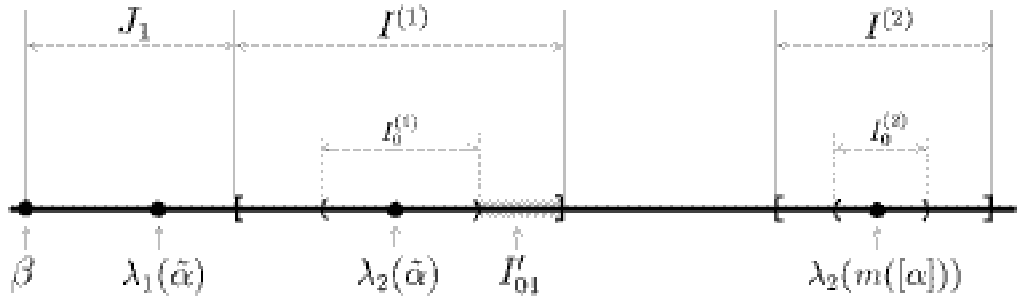

Figure.1 Eigenvalue enclosing and excluding

6 Computation of the Jacobian

In the Newton's iteration, we need to enclose the Jacobian matrix appearing in Step 3 in the algorithm. This can be done as follows.

Consider the i-th eigenvalue λ_i and corresponding eigenfunction u_i satisfying the following normalized eigenequation:

$$\begin{cases} -\Delta u_i + q(\alpha)u_i &= \lambda_i u_i, \\ \displaystyle\int_\Omega u_i^2 &= 1. \end{cases} \tag{8}$$

Then the pair $(u_i, \lambda_i) \equiv (u_i(\alpha), \lambda_i(\alpha))$ can be considered as a map from R^M into V. Differentiating formally both sides of (8) in α_j, we have

$$\begin{cases} -\Delta u_i' + \phi_j u_i + q(\alpha)u_i' &= \lambda_i' u_i + \lambda_i u_i', \\ \displaystyle\int_\Omega 2u_i u_i' &= 0, \end{cases} \tag{9}$$

where u_i' and λ_i' stand for $\dfrac{\partial u_i}{\partial \alpha_j}$ and $\dfrac{\partial \lambda_i}{\partial \alpha_j}$, respectively.

For the moment, we suppose the existence of derivatives u_i' and λ_i', which will be proved later.

Setting $v := u_i'$ and $\mu := \lambda_i'$, the equation (9) is written as

$$\begin{cases} -\Delta v + (q - \lambda_i)v &= (\mu - \phi_j)u_i, \\ \displaystyle\int_\Omega 2u_i v &= 0. \end{cases} \tag{10}$$

By calculating the inner product of both sides of the first equation in (10) with u_i, we have

$$(-\Delta v, u_i) + ((q - \lambda_i)v, u_i) = ((\mu - \phi_j)u_i, u_i). \tag{11}$$

where $(\phi, \psi) \equiv \displaystyle\int_\Omega \phi\psi dx$.

Therefore, using the integration by part taking account of H^2 regularity of the eigenfunction, we can easily obtain from (11)

$$\mu = \int_\Omega \phi_j u_i^2 dx, \tag{12}$$

that is, $\dfrac{\partial \lambda_i}{\partial \alpha_j} = \displaystyle\int_\Omega \phi_j u_i^2 dx, \quad 1 \le i, j \le M.$

Now, we consider the existence of a solution (v, μ) of (10), when (λ_i, u_i) is given. In order to prove this, we take a numerical approach which is based on the following theorem.

Theorem 2. If $(\phi, \gamma) \equiv (0, 0)$ is the unique solution of the homogeneous problem

$$\begin{cases} -\Delta \phi + (q - \lambda_i)\phi &- \gamma u_i = 0, \\ \displaystyle\int_\Omega 2u_i \phi &= 0, \end{cases} \tag{13}$$

then there exists a unique solution (v, μ) for the problem (10).

Proof. Let $(-\Delta)^{-1}$ denote the solution operator for the Poisson equation with homogeneous Diriclet boundary condition on Ω. We define the linear operator Φ on V by

$$\Phi \begin{pmatrix} \phi \\ \gamma \end{pmatrix} \equiv \begin{pmatrix} \phi \\ \gamma \end{pmatrix} - \begin{pmatrix} -(-\Delta)^{-1}(q-\lambda_i)(\cdot) & (-\Delta)^{-1}u_i(\cdot) \\ \int_\Omega 2u_i(\cdot)dx & 1 \end{pmatrix} \begin{pmatrix} \phi \\ \gamma \end{pmatrix}. \tag{14}$$

Then, since the second term of Φ is a compact operator on V, Φ is a Fredholm operator with index 0 on the same space. Therefore, the conclusion follows by the well known Fredholm alternative theorem. $\square$

We note that the hypothesis in Theorem 2. can be numerially proved by the similar method to that for the eigenvalue excluding technique in the previous section. Usually, the assertion in the theorem is proved for the interval potential function $q([\alpha])$ and corresponding set of all i-th eigenvalues $\lambda_i([\alpha])$ and eigenfunctions $u_i([\alpha])$. Although Theorem 2. plays an essential role on the existence of the Jacobian, we need further consideration below. Namely, we have to show the Jacobian J actually exists and each element of J coincides with the solution μ of (10), whenever Theorem 2. holds for each α in a certain interval $[\alpha]$.

For simplicity of notations, in stead of the problem (8) and its differential form (10), we consider the following eigenvalue problem for (u, λ) and associated linear equation in (v, μ) with potential $q \equiv q(\alpha) \in C^0(\bar{\Omega})$ which is differentiable in one real parameter α.

$$\begin{cases} -\Delta u + q(\alpha)u &= \lambda u, \\ \int_\Omega u^2 &= 1, \end{cases} \tag{15}$$

and, for an eigenpair (u, λ) of (15),

$$\begin{cases} -\Delta v + (q-\lambda)v &= (\mu - q')u, \\ \int_\Omega 2uv &= 0. \end{cases} \tag{16}$$

Here, in (16), we suppressed the dependency on the parameter α of u, λ, q and q'. The following theorem with one dimensional parameter will be sufficient for our present purpose.

Theorem 3. Let $[\alpha] \in IR$ be an interval, and let $(u(\alpha), \lambda(\alpha))$ be an eigenpair of (15) for $\alpha \in [\alpha]$. Assume that associated problem (16) has a unique solution pair (v, μ) for each $\alpha \in [\alpha]$. Then, the following limit properties hold:

$$\lim_{\delta \to 0} \frac{u(\alpha+\delta) - u(\alpha)}{\delta} = v \quad \text{in } H_0^1, \quad \lim_{\delta \to 0} \frac{\lambda(\alpha+\delta) - \lambda(\alpha)}{\delta} = \mu \quad \text{in } R. \tag{17}$$

Proof. First, we set

$$\begin{aligned} \phi \equiv \phi(\alpha, \delta) &:= \frac{u(\alpha+\delta) - u(\alpha)}{\delta} - v, \\ \rho \equiv \rho(\alpha, \delta) &:= \frac{\lambda(\alpha+\delta) - \lambda(\alpha)}{\delta} - \mu, \\ \sigma \equiv \sigma(\alpha, \delta) &:= \frac{q(\alpha+\delta) - q(\alpha)}{\delta} - q'. \end{aligned}$$

Then by some elementary calculations, we have

$$\left\{ \begin{array}{rcl} -\Delta\phi + (q(\alpha+\delta) - \lambda(\alpha+\delta))\phi &=& (\rho - \sigma)u(\alpha) \\ &+& \{(\lambda(\alpha+\delta) - \lambda(\alpha)) - (q(\alpha+\delta) - q(\alpha))\}v, \\ \displaystyle\int_\Omega (u(\alpha+\delta) + u(\alpha))\phi dx &=& \displaystyle -\int_\Omega (u(\alpha+\delta) - u(\alpha)) v dx. \end{array} \right. \tag{18}$$

Next, we fix an $\alpha \in [\alpha]$. Taking account of the interval property in Remark 7.2 in the next section, it can be considered that $u(\alpha+\delta) + u(\alpha) \in 2u([\alpha])$. Therefore, by the assumption, (18) has a unique solution pair $(\phi, \rho) \in V$ for each δ satisfying $\alpha + \delta \in [\alpha]$. Then, our aim is to show that $\delta \to 0$ implies $(\phi, \rho) \to (0, 0)$. In order to prove it, we consider the following slightly general non-homogeneous problem instead of (18):

$$\left\{ \begin{array}{rcl} -\Delta\phi + (q(\alpha+\delta) - \lambda(\alpha+\delta))\phi &=& \rho u(\alpha) + \xi, \\ \displaystyle\int_\Omega (u(\alpha+\delta) + u(\alpha))\phi &=& \eta. \end{array} \right. \tag{19}$$

Here, (ξ, η) is an element in $L^2(\Omega) \times R$.
We now define the linear map A_δ on V by

$$A_\delta \begin{pmatrix} \phi \\ \rho \end{pmatrix} := \begin{pmatrix} -(-\Delta)^{-1}(q(\alpha+\delta) - \lambda(\alpha+\delta))\phi + \rho(-\Delta)^{-1}u(\alpha) \\ \displaystyle\int_\Omega (u(\alpha+\delta) + u(\alpha))\phi dx + \rho \end{pmatrix}. \tag{20}$$

Then, (19) can be rewritten of the form

$$(I - A_\delta)z = \quad b, \tag{21}$$

where, $z \equiv {}^t(\phi, \rho)$, $b \equiv {}^t((-\Delta)^{-1}\xi, \eta)$ and I means the identity map on V.
It is now sufficient to show that if $b \to 0$ then $z \to 0$ uniformly in δ. Since, by our assumption, there exists the bounded inverse map $(I - A_\delta)^{-1}$ for each δ such that $\alpha+\delta \in [\alpha]$ from the Banach bounded inverse theorem, we prove that they are uniformly bounded in δ.

If such a conclusion does not hold, then there exists a positive constant K_0 , a sequence $\{\tau_n\}$ in V and positive numbers $\{\delta_n\}$ such that $\tau_n \to 0$, $\alpha + \delta_n \in [\alpha]$ and

$$||(I - A_{\delta_n})^{-1}\tau_n|| \geq K_0. \tag{22}$$

Setting $z_n := (I - A_{\delta_n})^{-1}\tau_n$, the sequence $\{z_n\}$ is bounded in V, because of the assumption and Remark 7.1 in Section 7. According to the weakly compactness of V, there exists an element $z_0 \in V$ such that a subsequence of $\{z_n\}$ weakly converges to it. We denote this subsequence by the same symbol. That is,

$$z_n \rightharpoonup z_0 \quad (n \to \infty).$$

On the other hand, since some subsequence $\{\delta_{n_j}\}$ of $\{\delta_n\}$ also converges to a real δ_0, $A_{n_j} \to A_0 := A_{\delta_0}$ in $\mathcal{L}(V, V)$. We denote again these subsequences by the same original symbols.
Now, by the compactness of the operator A_0, we have $A_0(z_n - z_0) \to 0$, strongly in V. Therefore, observe that

$$A_n z_n - A_0 z_0 = (A_n - A_0) z_n + A_0(z_n - z_0) \to 0 \quad \text{in} \quad V. \tag{23}$$

Thus, on the equality: $z_n = A_n z_n + \tau_n$, the left-hand side weakly converges to z_0, while the right-hand side strongly converges to $A_0 z_0$ in V from (23). Hence, by the usniqueness of the limit, we have $A_0 z_0 = z_0$ and the convergence has to be strong, i.e., $z_n \to z_0$ (strongly in V). This yields $z_0 = 0$ because of the uniqueness of the trivial solution for $(I - A_0)z = 0$. Therefore, we have $||z_n|| \to 0$, which contradicts with (22). Now, the assertion of the therorem immediately follows by the uniform boundedness of $(I - A_\delta)^{-1}$ in δ. $\Box$

Remark 6.1. If we use the verification scheme based on the Banach fixed point theorem as in [15] for the uniqueness proof of the solution of (13), the uniform boundedness of $(I - A_\delta)^{-1}$ will be derived from some consideration on the cotractivity condition around the trivial solution instead of the above compactness arguements.

We assumed here the continuous dependency of the eigenvalues and eigenfunctions on the parameter. But it can also be proved from the numerical results. That is, we have the following theorem.

Theorem 4. Under the same condition in the theorem 3, $\lambda(\alpha)$ and $u(\alpha)$ are locally continuous in α on $[\alpha]$.

Proof. For a fixed α, let $\phi \equiv \phi(\alpha, \delta) := u(\alpha+\delta) - u(\alpha)$, $\rho \equiv \rho(\alpha, \delta) := \lambda(\alpha+\delta) - \lambda(\alpha)$, and $\sigma \equiv \sigma(\alpha, \delta) := q(\alpha+\delta) - q(\alpha)$. Then we have

$$\begin{cases} -\Delta\phi + (q(\alpha+\delta) - \lambda(\alpha+\delta))\phi &= (\rho - \sigma)u(\alpha), \\ \displaystyle\int_\Omega (u(\alpha+\delta) + u(\alpha))\phi dx &= 0. \end{cases} \tag{24}$$

Hence, it is easily seen that if the λ is continuous at α then, by the similar arguments in the proof of Theorem 3, then the u is.

We now consider the continuity of λ. Observe that, taking the inner product with $u(\alpha+\delta)$,

$$(-\Delta\phi, u(\alpha+\delta)) + (q(\alpha+\delta) - \lambda(\alpha+\delta))\phi, u(\alpha+\delta))$$

$$= \rho(u(\alpha), u(\alpha+\delta)) - (\sigma u(\alpha), u(\alpha+\delta)).$$

Since the lefthand side vanishes, we have

$$\rho(u(\alpha), u(\alpha+\delta)) = (\sigma u(\alpha), u(\alpha+\delta)). \tag{25}$$

We now it would be possible, by the verified computational results, to deduce that there exists some constant C, independent of δ, satisfying

$$(u(\alpha), u(\alpha+\delta)) \geq C > 0 \quad \text{for arbitrary } \delta \text{ such that } \alpha+\delta \in [\alpha].$$

Then the continuity of the λ follows from (25). □

7 Enclosing the trivial solution of the linearlized problem with uniqueness

In this section, we mention about the method to enclose the trivial solution of (13) with uniqueness property. This can be done by applying our numerical verification method to the following set of equations with interval parameters $[\alpha]$.

$$\begin{cases} -\Delta\phi + (q([\alpha]) - \lambda([\alpha]))\phi \quad - \quad \gamma u([\alpha]) = 0, \\ \displaystyle\int_\Omega 2u([\alpha])\phi \quad = \quad 0, \end{cases} \tag{26}$$

where we suppressed the i-th dependency of eigenvalue and eigenfunction. We define the set valued map, similar to that in the proof of Theorem 3, which can be considered as the set of the operators by

$$A_{[\alpha]}\begin{pmatrix}\phi \\ \gamma\end{pmatrix} := \begin{pmatrix} -(-\Delta)^{-1}(q([\alpha]) - \lambda([\alpha])\phi + \gamma(-\Delta)^{-1}u([\alpha]) \\ \displaystyle\int_\Omega 2u([\alpha])\phi dx + \gamma \end{pmatrix}. \tag{27}$$

Then, (26) can be rewritten of the form

$$(I - A_{[\alpha]})z = \quad 0, \tag{28}$$

where, $z \equiv {}^t(\phi, \gamma)$. Let $S_N \subset V$ be a finite dimensional subspace with $\dim S_N = N$ and let P_N be the orthogonal projection from V onto S_N. Then the Newton-like operator on V is defined by

$$\mathcal{N}(z) := z - [I - A_{[\alpha]}]_N^{-1} P_N((I - A_{[\alpha]})z - \tau),$$

and set

$$T([\alpha], \tau)z := P_N\mathcal{N}(z) + (I - P_N)(A_{[\alpha]}z + \tau). \tag{29}$$

Here, τ is a small pertubation of the righthand side of (28) which would be used in later, and $[I - A_{[\alpha]}]_N^{-1}$ stands for the finite dimensional Newton-like operator on S_N for the problem, i.e., approximate inverse operator, see [7], [13], [14], [8] etc. for details. Then, the criterion for the proof of unique existence of the trivial solution can be presented as follows(cf. [9] or [4]):

Lemma 1. For a nonempty, bounded, closed and convex subset U in V, let assume that

$$T([\alpha], 0)U \overset{\circ}{\subset} U. \tag{30}$$

Then, the trivial solution is the unique solution of $(I - A_\alpha)z = 0$ for each $\alpha \in [\alpha]$, more exactly, for each equation (26) so that $u([\alpha])$ in (27) is replaced by the arbitrary $v \in u([\alpha])$, as described in the Remark 7.2 later.

The proof of this lemma is quite analogous to that in [9].
Usually, the set U satisfying the condition (30) is chosen such that $U = U_N \oplus U_\perp$, where $U_N \subset S_N$ and $U_\perp \subset S_N^\perp$.

And, actually, the verification condition (30) is taken of the form:

$$\begin{cases} P_N \mathcal{N}(U) & \overset{\circ}{\subset} \ U_N \\ \\ (I - P_N) A_{[\alpha]}(U) & \overset{\circ}{\subset} \ U_\perp. \end{cases} \tag{31}$$

Here, for an operator K, $K(U) \equiv \{Ku \mid u \in U\}$. The set U_N is chosen as a linear combination of base functions in S_N with interval coefficients, while the set $U_\perp$ a ball in $S_N^\perp$ with radius $\kappa \geq 0$.
Namely,

$$U_N = \{\tilde{\phi} \in S_N \mid \tilde{\phi} = \sum_{j=1}^{N} [\underline{A}_j, \overline{A}_j] \tilde{\phi}_j\},$$

and

$$U_\perp = \{\phi \in S_N^\perp \mid \ ||\phi||_V \leq \kappa\},$$

respectively, where $\{\tilde{\phi}_j\}_{j=1}^N$ is a basis of S_N.

Note that $P_N \mathcal{N}(U)$ can be directly computed or enclosed by solving a linear system of equations with interval righthand side (e.g., see [7], [14] for details). On the other hand, $(I - P_N) A_{[\alpha]}(U)$ is evaluated by using the constructive error estimates for the projection P_N of the form, for $z = (z_1, z_2) \in (H_0^1 \cap H^2) \times R$,

$$||((I - P_N)z)_1||_{H_0^1} \leq C(N) ||\Delta z_1||_{L^2}, \tag{32}$$

where $C(N)$ is a positive constant numerically determined. Here, $((I - P_N)z)_1 \in H_0^1$ stands for the first component of the element, and usually $((P_N)z)_2 = z_2$, therefore, $((I - P_N)z)_2 = 0$.
Thus, the former condition in (31) is validated as the inclusive relations of corresponding coefficient intervals and the latter part can be confirmed by comparing two nonnegative real numbers which correspond to the radii of balls. In the actual computations, we use an iterative method for both part in (31), which can be considered as an interval Newton-like method for the first part and the simple iteration for the second part(see [7], [13], [14] etc. for details).

Remark 7.1. Note that if the verification condition (30) is satisfied for a set U, then, from the continuity arguement on the operator $A_{[\alpha]}$, for each τ such that $||\tau||_V \leq \varepsilon$ for sufficiently small ε,

$$T([\alpha],\tau)U \overset{\circ}{\subset} U \tag{33}$$

holds. This fact implies that each unique solution for small perturbed non-homogeneous equations of (28) also belongs to the same set $\overset{\circ}{U}$.

Remark 7.2. We give some consideration on the inclusion property : for $\alpha,\ \ \alpha+\delta \in [\alpha]$,

$$u(\alpha+\delta)+u(\alpha) \in 2u([\alpha]). \tag{34}$$

Notice that $u([\alpha])$ is treated, in the actual computational process, as the set of elements in H_0^1 of the form

$$u([\alpha]) = \sum_{j=1}^{N} B_j\tilde{\phi}_j \oplus < \kappa >,$$

where $< \kappa > := \{\phi \in (S_N)_1^{\perp} \subset H_0^1 \mid \ ||\phi||_{H_0^1} \leq \kappa\}$ where $(S_N)_1^{\perp}$ means the orthogonal complement subspace in H_0^1 for the first compoment of S_N. By the fact that both of $u(\alpha)$ and $u(\alpha+\delta)$ belong to $u([\alpha])$, there exist $b_j^{(i)} \in B_j$ and $\kappa_i \in < \kappa >$, for $i = 1, 2$, such that

$$u(\alpha) = \sum_{j=1}^{N} b_j^{(1)}\tilde{\phi}_j \oplus \kappa_1 \quad \text{and} \quad u(\alpha+\delta) = \sum_{j=1}^{M} b_j^{(2)}\tilde{\phi}_j \oplus \kappa_2,$$

which yields that

$$b_j^{(2)} + b_j^{(1)} \in 2B_j,\ 1 \leq j \leq M, \quad \text{and} \quad ||\kappa_1+\kappa_2||_{H_0^1} \leq 2\kappa.$$

Thus the desired inclusion (34) follows.

8 Numerical examples

We present several numerical examples which show our enclosing algorithm really works. Let $\Omega := (0,\pi)\times(0,\pi)$ in R^2. In order to avoid the non-deterministic property, we assume that the potential function has the following symmetry about the barycenter of Ω:

$$q(x,y) = q(\pi-x,\pi-y).$$

We assumed the potential function q of the form(M=4):
$q(x,y) := \alpha_1 + \alpha_2 \cos(2x) + \alpha_3 \cos(2y) + \alpha_4 \cos(2x)\cos(2y)$.
The spectral approximation and the constructive error estimates were utilized for the enclosing eigenvalues, eigenfunctions and Jacobian as well as excluding eigenvalues. That is, $S_N \subset V$ is taken as

$$S_N := \{(\tilde{\phi}, \gamma) \in V \mid \tilde{\phi}(x,y) = \sum_{i=1}^{K_1} \sum_{j=1}^{K_2} A_{ij} \sin(ix)\sin(jy), \quad A_{ij} \in R,\ \gamma \in R\}.$$

The orthogonal projection $P_N : V \longrightarrow S_N$ is defined by, for $u = (u_1, u_2) \in V$,

$$(\nabla(u_1 - (P_N u)_1), \nabla\tilde{\phi}_N) + (u_2 - (P_N u)_2, \gamma) = 0, \quad \forall(\tilde{\phi}_N, \gamma) \in S_N,$$

where the second inner product implies the simple multiplication of two real numbers. Then, it is known that the constant $C(N)$ in (32) can be taken as

$$C(N) = \frac{\sqrt{2}(N+1)}{(N+1)^2 + 1}.$$

Example 1:

We tried to reconstruct $q(x,y) = \cos(2x)$ from four eigenvalues. We took initial guess: $q^{(0)}(x,y) = 0.9\cos(2x)$.
By using the above $q^{(0)}$, we first executed the approximate Newton process like as the algorithm in Section 3(cf. [11]) to refine the inital guess. As a result, we obtained an approximation which is very closed to $\cos(2x)$, within almost machine epsilon, and adopted it as $\hat{q}$.
Prescribed eigenvalues are as follows:
$\mu_1 = 1.470654354933839, \quad \mu_2 = 4.470654354933839,$
$\mu_3 = 4.979189215751357, \quad \mu_4 = 7.979189215751357.$
The above data was obtained as the lowest 4 eigenvalues from the approximate solutions of the ordinary eigenvalue problem with $q(x,y) = \cos(2x)$ discretized by using a spectral Galerkin method in S_N.
We succeeded the reconstruction of the potential function $q([\alpha])$ with local uniqueness as the following results:

$$\begin{array}{lll} \alpha_1 \in & & [-0.25209635682266250, 0.25209635682266260] \times 10^{-9} \\ \alpha_2 \in & 1+ & [-0.13274480437239270, 0.13274480437239280] \times 10^{-8} \\ \alpha_3 \in & & [-0.96398812017349510, 0.96398812017349600] \times 10^{-9} \\ \alpha_4 \in & & [-0.50884999871818090, 0.50884999871818120] \times 10^{-8} \end{array} \tag{35}$$

Here, we used approximation subapace S_N with $K_1 = K_2 = 10$.
Then, for the above interval potential function $q([\alpha])$, corresponding eigenvalues are enclosed as

$$\lambda_1([\alpha]) \in [0.147065435448617, 0.147065435538151] \times 10^1,$$
$$\lambda_2([\alpha]) \in [0.447065435444676, 0.447065435542092] \times 10^1,$$
$$\lambda_3([\alpha]) \in [0.497918921557205, 0.497918921593066] \times 10^1,$$
$$\lambda_4([\alpha]) \in [0.797918921555445, 0.797918921594826] \times 10^1.$$

Furthermore, in the verification of the existence of the Jacobian, we could also enclose the trivial solutions in the small intervals and norms with order of magnitude $\approx 10^{-9} - 10^{-12}$.

And, we successfully completed the eigenvalue excluding process between these eigenvalues for the midpoint potential function $q(m[\alpha])$. In this procedure, we needed some addtional computations with more fine approxiation subspaces, because the uniquely existential condition is fairly delicate problem at the points very closed to the actual eigenvalues. Actually, it was necessary to take so that $K := K_1 = K_2 > 10$, e.g., maximum case $K = 50$.

Finally, in order to examine the sensitivity for the input data $\{\mu_i\}_{i=1,\cdots,M}$, we tried to add some small perturbed intervals to the original point data, such as $\mu_i := \mu_i + [-\varepsilon, +\varepsilon]$. As the result, we succeeded the reconstruction for $\varepsilon \le 0.05$. But, for $\varepsilon = 0.1$, the eigenpair enclosing process in Section 4 failed due to the wide width of intervals, which would be considered rather natural by the property of interval arithmetic.

Example 2: Reconstruction of $q(x, y) = -2 + \cos(2x) - \cos(2x)\cos(2y)$

The way of setting for $\hat{q}$ is same as previous example. $K_1 = K_2 = 10$ for S_N. Prescribed eigenvalues are
$\mu_1 = -0.8270142949509, \quad \mu_2 = 2.4615199904499,$
$\mu_3 = 2.9500526665779, \quad \mu_4 = 5.9765623708221.$
Enclosing intervals α_j, $1 \le j \le 4$, with uniqueness are as follows:

$$\begin{array}{lcl} \alpha_1 \in & -2 & +[-0.3089173380, 0.3089173380] \times 10^{-8} \\ \alpha_2 \in & 1 & +[-0.1490897049, 0.1490897049] \times 10^{-7} \\ \alpha_3 \in & & +[-0.2416657724, 0.2416657724] \times 10^{-7} \\ \alpha_4 \in & -1 & +[-0.1037820952, 0.1037820952] \times 10^{-6} \end{array} \qquad (36)$$

Example 3: Reconstruction of $q(x,y) = 10 + 5\cos(2x) + 3\cos(2y) + 3\cos(2x)\cos(2y)$

The way of setting for $\hat{q}$ is same as example 1 and $K_1 = K_2 = 15$. Prescribed eigenvalues are

$\mu_1 = 8.9232011202383, \quad \mu_2 = 11.9106924233670,$

$\mu_3 = 13.6562169567569, \quad \mu_4 = 16.7797156194290.$

Enclosing intervals α_j, $1 \leq j \leq 4$, with uniqueness are as follows:

$$\begin{array}{lcl} \alpha_1 & \in & 10 + [-0.4072392255, 0.4072392255] \times 10^{-8} \\ \alpha_2 & \in & 5 + [-0.5651032512, 0.5651032512] \times 10^{-8} \\ \alpha_3 & \in & 3 + [-0.4740982608, 0.4740982608] \times 10^{-8} \\ \alpha_4 & \in & 3 + [-0.6225533666, 0.6225533666] \times 10^{-8} \end{array} \tag{37}$$

We used the following computing facility in Computing and Communications Center, Kyushu University to execute the above enclosing algorithm:

- Fujitsu GP7000F Model 900(SPARC64-GP: 300MHz)

- Fujitsu Fortran Compiler Driver Version 4.0.2

- INTLIB-90(Interval Computation Module)

References

[1] Alefeld, G., On the convergence of some interval-arithmetic modifications of Newton's method, SIAM J. Numer. Anal. 21(1984), 363-372.

[2] Kirsch, A., An introduction to the mathematical theory of inverse problems, Springer, New York, 1996.

[3] Lohner, R., Enclosing the solutions of ordinary initial and boundary value problems, Computerarithmetic (eds. Kaucher, E. et al.), B.G. Teubner Stuttgart, (1987), 255-286.

[4] Nagatou, K., Yamamoto, N. & Nakao, M.T., An approach to the numerical verification of solutions for nonlinear elliptic problems with local uniqueness, Numerical Functional Analysis and Optimization **20** (1999), 543-565.

[5] Nagatou, K., A numerical method to verify the elliptic eigenvalue problems including a uniqueness property, *Computing* **63** (1999), 109-130.

[6] Nakao, M.T., A numerical verification method for the existence of weak solutions for nonlinear boundary value problems, Journal of Mathematical Analysis and Applications 164 (1992), 489-507.

[7] —, Solving nonlinear elliptic problems with result verification using an H^{-1} residual iteration, Computing Suppl. 9 (1993), 161-173.

[8] Nakao, M.T. & Yamamoto, N., Numerical verification of solutions for nonlinear elliptic problems using L^∞ residual method, Journal of Mathematical Analysis and Applicatons **217**, (1998), 246-262.

[9] Nakao, M.T., Yamamoto, N. & Nagatou, K., Numerical verifications of eigenvalues of second-order elliptic operators, Japan Journal of Industrial and Applied Mathematics 16 (1999), 307-320.

[10] Neher, M., Ein Einschließungsverfahren für das inverse Dirichletproblem, Dissertation, University of Karlsruhe, 1993.

[11] —, Enclosing solutions of an inverse Sturm-Liouville problem with finite data, *Computing* 53 (1994), 379-395.

[12] Rump, S.M., Solving algebraic problems with high accuracy, A new approach to scientific computation (eds. Kulisch, U. & Miranker, W.L.), Academic Press, New York, 1983.

[13] Watanabe, Y. & Nakao, M.T., Numerical verifications of solutions for nonlinear elliptic equations, Japan Journal of Industrial and Applied Mathematics 10 (1993), 165-178 .

[14] Yamamoto, N. & Nakao, M.T., Numerical verifications for solutions to elliptic equations using residual iterations with higher order finite element, Journal of Computational and Applied Mathematics 60 (1995), 271-279.

[15] Yamamoto, N., A numerical verification method for solutions of boundary value problems with local uniqueness by Banach's fixed point theorem, SIAM J. Numer. Anal. 35 (1998), 2004-2013.