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Constructive L^2 Error Estimates for Finite Element Solutions of the Stokes Equations

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Abstract. Constructive L^2 error estimates for finite element solutions of the Stokes equations are described. We show that the L^2 error bounds for the velocity can be obtained in a posteriori and explicit a priori sense. Some numerical examples which confirm us the expected rates of convergence are presented.

Keywords: Stokes equations, guaranteed error bounds, computable error

1. Introduction

Using the numerical estimates of a constant related to the so-called inf-sup condition, we proposed, in [11] and [12], a method to estimate the guaranteed a posteriori H_0^1 error bounds of the finite element solutions for the Stokes problem in mathematically rigorous sense. Furthermore, these papers describe a method to derive the constructive H_0^1 a priori error estimates based on the estimation of the largest eigenvalues for related matrices. These results are confirmed by some numerical examples.

On the other hand, in many cases, the L^2 error estimates are obtained by some duality method called as Aubin-Nitsche's trick in the mathematical theory of finite element methods(e.g.,[4], [14]). And the L^2 rate of convergence generally has a higher order rate than the H^1 error. This process sometimes is referred to as "L^2 lifting". However, there is no such result for the Stokes problems in the constructive sense up to now. Particularly, it is not clear if the expected optimal L^2 rates of converegence could be attained for the actual numerical computations.

In this paper, we clarify that an Aubin-Nitsche-like technique can also be applied to the constructive L^2 error estimates and establish the estimates both in a posteriori and a priori sense by using the results obtained in our previous works. Furthermore, by illustrating some numerical examples, we will show that the expected optimal rate of convergence for the L^2 error are actually attained.

Also, we notice that our approach is essentially different from the existing literatures by [1], [16] etc. which give only an error indicator rather than the guaranteed error bounds. That is, in their works, the explicit value of constants in the error estimates are not resolved at all. Another difference is that our work is deeply related to the numerical verification or the numerical existence proof of exact solutions for associated nonlinear problems, i.e.,the Navier-Stokes equations([17]).

2. Stokes problem

We consider the following Stokes problem

$$\begin{cases} -\nu\Delta u + \nabla p = f & \text{in} \quad \Omega, \\ \operatorname{div} u = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where $\nu > 0$ is the viscosity constant, $u = (u_1, u_2)^T$ the two-dimensional velocity field, $f = (f_1, f_2)^T$ a pair of L^2 function on Ω which means a density of body forces per unit mass and Ω a convex polygonal domain in $\mathbb{R}^2$. Here, p represents a kinematic pressure field and $\operatorname{div} u = 0$ means the incompressibility condition.

We denote by $H^k(\Omega)$ the usual k-th order Sobolev space on Ω, and define $(\,\cdot,\cdot\,)$ as the inner product in $L^2(\Omega)$ and put

$$\begin{aligned} H_0^1(\Omega) &\equiv \{v \in H^1(\Omega)\ ;\ v = 0 \ \text{ on } \ \partial\Omega\}, \\ L_0^2(\Omega) &\equiv \{v \in L^2(\Omega)\ ;\ (\,v, 1\,) = 0\}, \\ \mathcal{S} &\equiv H_0^1(\Omega)^2 \times L_0^2(\Omega). \end{aligned}$$

The norm in $L^2(\Omega)$ and $H_0^1(\Omega)$ is denoted by $|q|_0 \equiv (\,q, q\,)^{1/2}$ and $|v|_1 \equiv |\nabla v|_0$, respectively. We also define $H^2(\Omega)$-seminorm $|\cdot|_2$ by

$$|u|_2 \equiv \left(\left|\frac{\partial^2 u}{\partial x^2}\right|_0^2 + 2\left|\frac{\partial^2 u}{\partial x \partial y}\right|_0^2 + \left|\frac{\partial^2 u}{\partial y^2}\right|_0^2 \right)^{1/2}.$$

In what follows, since no confusion may arise, we will use the same notations for the corresponding norms and inner products in $L^2(\Omega)^2$ and $H_0^1(\Omega)^2$ as in $L^2(\Omega)$ and $H_0^1(\Omega)$, respectively.

We introduce a bilinear form $\mathcal{L}$ on $\mathcal{S} \times \mathcal{S}$ by

$$\mathcal{L}(\,[u,p],[v,q]\,) \equiv \nu(\,\nabla u, \nabla v\,) - (\,p, \operatorname{div} v\,) - (\,q, \operatorname{div} u\,) \qquad [u,p],\ [v,q] \in \mathcal{S}. \tag{2}$$

Then, the standard variational formulation of (1) (cf.[5])is given by

$$\begin{aligned} &\text{find } \ [u,p] \in \mathcal{S} \ \text{ such that} \\ &\mathcal{L}(\,[u,p],[v,q]\,) = (\,f, v\,) \qquad \forall [v,q] \in \mathcal{S}. \end{aligned} \tag{3}$$

It is well-known(cf.[5]) that there exists a constant $\beta > 0$ depending only on Ω such that for all $q \in L_0^2(\Omega)$, there exists a $v \in H_0^1(\Omega)^2$ satisfying

$$\operatorname{div} v = q, \qquad |v|_1 \le \frac{1}{\beta}|q|_0.$$

Here, β is a constant related to the inf-sup condition for $\mathcal{L}$ which assures that the problem (3) has a unique solution in $\mathcal{S}$. For the star shaped domains, by Horgan([7]), this constant β can be numerically determined, for example, $1/\beta^2 \le 4 + 2\sqrt{2}$ for the square.

3. A posteriori and a priori H_0^1 error estimates

Let $\mathcal{T}_h$ be a family of triangulations of $\Omega \subset \mathbb{R}^2$, which consist of triangles or quadrilaterals dependent on a scale parameter $h > 0$. For $\mathcal{T}_h$, we denote by $X_h \subset H_0^1(\Omega) \cap C(\bar{\Omega})$ and $Y_h \subset L_0^2(\Omega) \cap C(\bar{\Omega})$ the finite element subspaces for the approximation of each component of the velocity u and the pressure p, respectively. The standard finite element solution $[u_h, p_h] \in X_h^2 \times Y_h$ to (3) is defined by

$$\mathcal{L}(\,[u_h, p_h], [v_h, q_h]\,) = (\,f, v_h\,) \qquad \forall [v_h, q_h] \in X_h^2 \times Y_h. \tag{4}$$

We denote by X_h^* a subspace of $H^1(\Omega)$ in which the basis of X_h^* is the union of the basis of X_h and the base functions corresponding to nodes on the boundary $\partial\Omega$. We also define P_0 as an L^2-projection from $L^2(\Omega)$ to X_h, $\hat{P}_0$ as an L^2-projection from $L^2(\Omega)$ to X_h^* and P_1 as an H_0^1-projection from $H_0^1(\Omega)$ to X_h, respectively. For each $w_h \in X_h$, we define $\overline{\nabla} w_h \in (X_h^*)^2$ and $\overline{\Delta} w_h \in L^2(\Omega)$ by

$$\begin{aligned} \overline{\nabla} w_h &\equiv (\hat{P}_0 \frac{\partial w_h}{\partial x}, \hat{P}_0 \frac{\partial w_h}{\partial y})^T, \\ \overline{\Delta} w_h &\equiv \operatorname{div} \overline{\nabla} w_h, \end{aligned}$$

respectively. We assume, as the approximation property of X_h, that

$$\inf_{\xi \in X_h} |v - \xi|_1 \le C_0 h |v|_2 \qquad \forall v \in H_0^1(\Omega) \cap H^2(\Omega),$$

where C_0 is a positive constant independent of v and h which can be numerically determined (see section 5). This assumption holds for many finite element subspaces (*cf.*[4]).

As is well-known, e.g. [5], (3) has a unique solution $[u, p]$. And we suppose that (4) has a solution $[u_h, p_h]$.

Then, we have the following a posteriori error estimates for finite element solutions of the Stokes equations ([12]).

THEOREM 1 (A POSTERIORI ERROR ESTIMATES) *Let $[u, p]$ and $[u_h, p_h]$ be solutions of (3) and (4), respectively. Then, the following a posteriori error estimates hold :*

$$\left\{ \begin{aligned} |u - u_h|_1 &\le \left(\frac{1}{\nu^2} + \frac{1}{\beta^2} \right)^{1/2} C(u_h, p_h), \\ |p - p_h|_0 &\le \left(\frac{1}{\beta} + \frac{\nu}{\beta^2} \right) C(u_h, p_h), \end{aligned} \right. \tag{5}$$

where $C(u_h, p_h)$ is an a posteriori error estimator which can be computed using the finite element solutions $[u_h, p_h]$ by

$$C(u_h, p_h) \equiv \nu|\overline{\nabla}u_h - \nabla u_h|_0 + C_0 h|\nu\overline{\Delta}u_h - \nabla p_h + f|_0 + |\mathrm{div}\, u_h|_0. \tag{6}$$

Next, we take the positive constants K_1 and K_2 such that

$$|\mathrm{div}\, u_h|_0 \le K_1|P_0 f|_0, \tag{7}$$

$$|-\nabla p_h + P_0 f|_0 \le K_2|P_0 f|_0, \tag{8}$$

independent of $f \in L^2(\Omega)$ (cf.[11]). In order to keep the present paper self-contained, we briefly describe how to estimate K_1 and K_2. Let us denote the basis of X_h as ϕ_j $(j = 1, \cdots, n,\ n = \dim X_h)$, $f = (f_1, f_2)^T$ and $g \in \mathbb{R}^{2n}$ as

$$g \equiv ((f_1, \phi_1), \cdots, (f_1, \phi_n), (f_2, \phi_1), \cdots, (f_2, \phi_n))^T.$$

Then, each term in (7) and (8) can be represented by quadratic forms of $2n$-dimensional vectors g as follows:

$$\begin{aligned} |\mathrm{div}\, u_h|_0{}^2 &= g^T A_1 g, \\ |-\nabla p_h + P_0 f|_0{}^2 &= g^T A_2 g, \\ |P_0 f|_0{}^2 &= g^T L g, \end{aligned}$$

where, A_1, A_2 are $2n \times 2n$ symmetric matrices, L a $2n \times 2n$ positive definite and symmetric matrix. Hence, the upper bounds of K_1 and K_2 can be estimated as follows.

$$K_1 \le \left(\sup_{x \in \mathbb{R}^{2n}} \frac{x^T A_1 x}{x^T L x} \right)^{1/2}, \quad K_2 \le \left(\sup_{x \in \mathbb{R}^{2n}} \frac{x^T A_2 x}{x^T L x} \right)^{1/2}.$$

Therefore, the estimation of these values is reduced to finding the maximum eigenvalue of the following generalized eigenvalue problem:

$$Ax = \lambda L x,$$

and using a procedure proposed by [18], we can estimate these eigenvalues.

Then, we can show the following a priori estimates ([12]).

Theorem 2 (a priori error estimates) *For all $f \in L^2(\Omega)^2$, under the assumption of Theorem 1, it holds that*

$$\begin{cases} |u - u_h|_1 \le \left(\dfrac{1}{\nu^2} + \dfrac{1}{\beta^2} \right)^{\frac{1}{2}} C(h)\,|f|_0, \\ |p - p_h|_0 \le \left(\dfrac{1}{\beta} + \dfrac{\nu}{\beta^2} \right) C(h)\,|f|_0. \end{cases} \tag{9}$$

where $C(h)$ is a positive constant which can be computed from K_1, K_2, h and C_0 by

$$C(h) \equiv \sqrt{(K_1 + C_0 h K_2)^2 + (C_0 h)^2}. \tag{10}$$

4. L^2 error estimate for the velocity

In this section, we present our main result of this paper. We show that we can derive an explicit bound on $|u-u_h|_0$ using the estimation $|u-u_h|_1$ and a method like well-known *Aubin-Nitsche trick* for Poisson's equations ([4], [14], [19]).

THEOREM 3 *Let $[u,p]$ and $[u_h,p_h]$ be solutions of (3) and (4), respectively. Then, the following estimates hold :*

$$|u-u_h|_0 \le \nu C_1 |u-u_h|_1 + C_2 |\mathrm{div}\, u_h|_0 + K_1 |p-p_h|_0, \tag{11}$$

where

$$\begin{aligned} C_1 &\equiv \left(\frac{1}{\nu^2}+\frac{1}{\beta^2}\right)^{\frac{1}{2}} C(h), \\ C_2 &\equiv \left(\frac{1}{\beta}+\frac{\nu}{\beta^2}\right) C(h). \end{aligned}$$

Proof: Let $[u,p]$ and $[u_h,p_h]$ be solutions of (3) and (4), respectively, and consider the Stokes equation

$$\left\{\begin{array}{rlll} -\nu\Delta\phi+\nabla\psi = & u-u_h & \text{in} & \Omega, \\ \mathrm{div}\,\phi = & 0 & \text{in} & \Omega, \\ \phi = & 0 & \text{on} & \partial\Omega. \end{array}\right. \tag{12}$$

¿From (12) and integration by parts, we have

$$\begin{aligned} (\,u-u_h, u-u_h\,) &= (\,u-u_h, -\nu\Delta\phi+\nabla\psi\,) \\ &= \nu(\,\nabla(u-u_h), \nabla\phi\,) + (\,\mathrm{div}\,u_h, \psi\,). \end{aligned} \tag{13}$$

Moreover, for any $v_h \in X_h^2$ and $q_h \in Y_h$, from (3) and (4) we have

$$\nu(\,\nabla(u-u_h), \nabla v_h\,) + (\,q_h, \mathrm{div}\,u_h\,) - (\,p-p_h, \mathrm{div}\,v_h\,) = 0, \tag{14}$$

and hence from (13), (14) and Schwarz's inequality,

$$\begin{aligned} {|u-u_h|_0}^2 &= \nu(\,\nabla(u-u_h), \nabla(\phi-v_h)\,) + (\,\psi-q_h, \mathrm{div}\,u_h\,) + (\,p-p_h, \mathrm{div}\,v_h\,) \\ &\le \nu|u-u_h|_1\,|\phi-v_h|_1 + |\mathrm{div}\,u_h|_0\,|\psi-q_h|_0 + |p-p_h|_0\,|\mathrm{div}\,v_h|_0 \end{aligned} \tag{15}$$

is obtained.

Now, taking $[v_h,q_h] \in X_h^2 \times Y_h$ as the finite element approximation of (12), Theorem 2 and (7) imply that

$$|\phi-v_h|_1 \le C_1|u-u_h|_0, \tag{16}$$

$$|\psi-q_h|_0 \le C_2|u-u_h|_0, \tag{17}$$

$$|\mathrm{div}\,v_h|_0 \le K_1|u-u_h|_0, \tag{18}$$

Consequently, from (15), (16), (17) and (18), we have the L^2 error estimates (11). ■

The right hand side of (11) can be a posteriori estimated by Theorem 1. Moverover, by virtue of Theorem 2 and (7), we obtain an a priori estimate as follows:

THEOREM 4 *For all $f \in L^2(\Omega)$, it holds that*

$$|u - u_h|_0 \le (\nu C_1^2 + 2C_2K_1)|f|_0. \tag{19}$$

Using Theorem 4, we can calculate the explicit a priori constant in the L^2 error bounds for finite element solutions of the velocity to the Stokes problem.

5. Numerical examples

Let Ω be a rectangular domain in $\mathbb{R}^2$ such that $\Omega = (0,1) \times (0,1)$. Also let $\delta_x : 0 = x_0 < x_1 < \cdots < x_L = 1$ be a uniform partition, and let δ_y be the same partition as δ_x for y direction. We define the partition of Ω by $\delta \equiv \delta_x \otimes \delta_y$. L denotes the number of partitions for the interval $(0,1)$, i.e. $h = 1/L$. Further, we define the finite element subspace X_h and Y_h by $X_h \equiv \mathcal{M}_0^2(x) \otimes \mathcal{M}_0^2(y)$ where $\mathcal{M}_0^2(x)$, $\mathcal{M}_0^2(y)$ are sets of continuous piecewise quadratic polynomials on $(0,1)$ under the above partition δ with homogeneous boundary condition. And set $Y_h \equiv (\mathcal{M}^1(x) \otimes \mathcal{M}^1(y)) \cap L_0^2(\Omega)$ where $\mathcal{M}^1(x)$, $\mathcal{M}^1(y)$ be piecewise linear on $(0,1)$. We set the constant $\nu = 1$. Then we can take $C_0 = 1/(2\pi)$ ([13]) and β as in the end of section 1.

As a numerical example, we consider some error estimates related to the residual form of the finite element approximation for the following stationary Navier-Stokes equations:

$$\left\{ \begin{array}{rcll} -\Delta u + \nabla p &=& -(u \cdot \nabla)u + f & \text{in} \quad \Omega, \\ \operatorname{div} u &=& 0 & \text{in} \quad \Omega, \\ u &=& 0 & \text{on} \quad \partial\Omega. \end{array} \right. \tag{20}$$

We take a finite element approximation $[u_h, p_h] \in X_h^2 \times Y_h$ to (20) satisfying

$$\mathcal{L}(\,[u_h, p_h], [v_h, q_h]\,) = -(\,(u_h \cdot \nabla)u_h, v_h\,) + (\,f, v_h\,) \qquad \forall [v_h, q_h] \in X_h^2 \times Y_h. \tag{21}$$

Then, $[u_h, p_h]$ coincides with the finite element solution to the Stokes equation

$$\left\{ \begin{array}{rcll} -\Delta \bar{u} + \nabla \bar{p} &=& -(u_h \cdot \nabla)u_h + f & \text{in} \quad \Omega, \\ \operatorname{div} \bar{u} &=& 0 & \text{in} \quad \Omega, \\ \bar{u} &=& 0 & \text{on} \quad \partial\Omega. \end{array} \right. \tag{22}$$

Now, we set $v_0 \equiv \bar{u} - u_h \in H_0^1(\Omega)^2$, then v_0 corresponds to the residual error for the velocity approximation u_h to (20) (cf.[10], [19]).

We now briefly describe how Theorem 3 and 4 are used in practice for our present purpose.
First, for various $L = 1/h$, we calculate the approximate solution $[u_h, p_h]$ to (20) by some appropriate method, e.g., Newton's method.
Next, we decide K_1 and K_2 in (7) and (8), respectively, by estimating the largest eigenvalues.
We also compute, by using the above u_h, $|\overline{\Delta} u_h - \nabla p_h - (u_h \cdot \nabla) u_h + f|_0$, $|\mathrm{div}\, u_h|_0$ and $|\overline{\nabla} u_h - \nabla u_h|_0$ in (6).
Then, using those estimates, positive constants $C(u_h, p_h)$ in (6) and $C(h)$ in (10) can be computed.
Finally, by the application Theorem 1, 2, 3 and 4, the numerical estimates of the H_0^1 and L^2 norms for v_0 can be obtained in both of the a posteriori and a priori sense.

We emphasize that these norm estimates play an important role for the numerical verification of the solution of the stationary Navier-Stokes equations (cf. [10], [17], [18], [19]). We choose the vector function f so that

$$\begin{aligned} u_1(x,y) &= \sin^2 \pi x \sin \pi y \cos \pi y \\ u_2(x,y) &= -\sin^2 \pi y \sin \pi x \cos \pi x \\ p(x,y) &= -\cos 2\pi x \cos 2\pi y / 16 \end{aligned}$$

are the exact solutions for (20). In this case, $||u_h||_\infty \approx 0.71$ and $|-(u_h \cdot \nabla) u_h + f|_0 \approx 292.3$, where $||\cdot||_\infty$ stands for the L^∞ norm on Ω.

Figure 1 and Figure 2 show the $L(= 1/h)$ dependency of $|\overline{\nabla} u_h - \nabla u_h|_0$ and $|\mathrm{div}\, u_h|_0$ which correspond to the first and third term in (6), respectively.

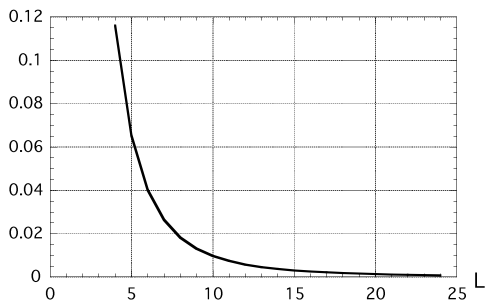

Figure 1. $|\overline{\nabla} u_h - \nabla u_h|_0$

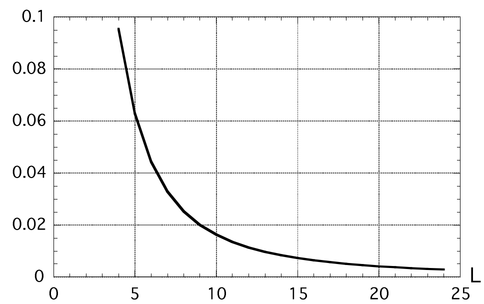

Figure 2. $|\mathrm{div}\, u_h|_0$

Figure 3 illustrates $|\overline{\Delta} u_h - \nabla p_h - (u_h \cdot \nabla) u_h + f|_0$, the second term in (6) divided by $C_0 h$.

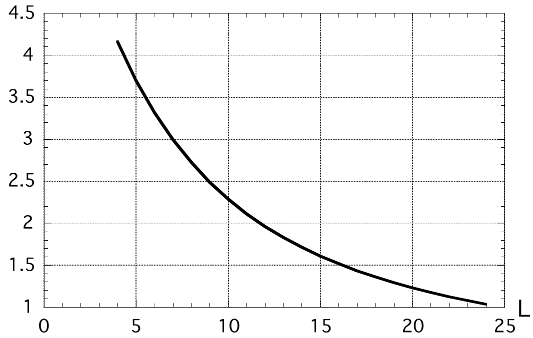

Figure 3. $|\overline{\Delta}u_h - \nabla p_h - (u_h \cdot \nabla)u_h + f|_0$

The dependency of $|\overline{\nabla}u_h - \nabla u_h|_0$ and $|\mathrm{div}\, u_h|_0$ seem to be almost of order $O(h^2)$, and that of $|\overline{\Delta}u_h - \nabla p_h - (u_h \cdot \nabla)u_h + f|_0$ to be almost of order $O(h)$.

Figure 4 illustrates $|v_0|_1$ and $|v_0|_0$ for various $L = 1/h$ using the a posteriori and a priori methods.

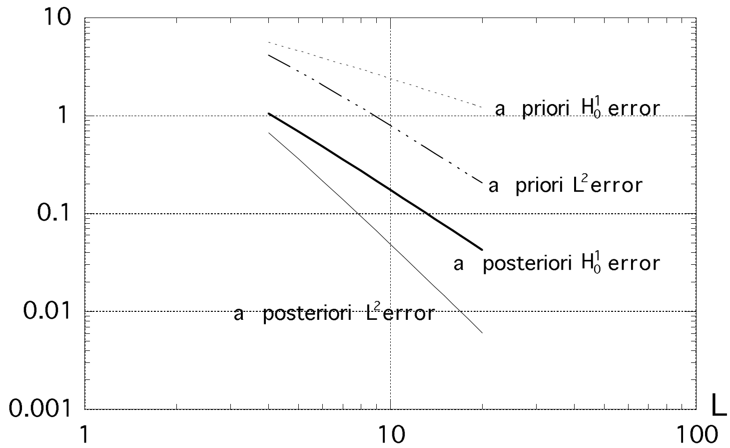

Figure 4. h-dependency of $|v_0|_1$ and $|v_0|_0$

These numerical examples show that the error estimates by Theorem 1-4 actually enable us the expected rates of convergence of errors, i.e., in the optimal sense as follows:

a priori estimate of $\lvert v_0\rvert_1$	$\sim$	$O(h^1)$
a priori estimate of $\lvert v_0\rvert_0$	$\sim$	$O(h^2)$
a posteriori estimate of $\lvert v_0\rvert_1$	$\sim$	$O(h^2)$
a posteriori estimate of $\lvert v_0\rvert_0$	$\sim$	$O(h^3)$

The numerical examples are computed on FUJITSU VP2600/10 vector processor by the usual computer arithmetic with double precision. So, the round-off errors in these examples are neglected. Namely, the results are not validated in the sense of precise interval arithmetic.

Remark. Constructive error estimates for the finite element solutions of differential equations essentially include the infinite dimensional aspect. Therefore, our main emphasis is put on the principle and the way to reduce such an infinite dimensional problem to the finite dimension, particularly, for the Stokes problem.

In that sense, the validation of the finite dimensional computation is considered as a separated problem from our main subject. Therefore, the above numerical results should be sufficient for our present purpose. Of course, in case that we need the rigorous mathematical proof, we should take account of errors arised from the finite dimensional problem by some verified approaches such as [3], [8], [9]etc.

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