A Numerical Verification of Bifurcated Solutions for the Heat Convection Problem

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1 Introduction

We consider the following two-dimensional (x-z) Oberbeck-Boussinesq equations for the Rayleigh-Bénard convection:

$$\begin{cases} u_t + uu_x + \omega u_z &= p_x + \mathcal{P}\Delta u, \\ \omega_t + u\omega_x + \omega\omega_z &= p_z - \mathcal{P}\mathcal{R}\,\theta + \mathcal{P}\Delta\omega, \\ u_x + \omega_z &= 0, \\ \theta_t + \omega + u\theta_x + \omega\theta_z &= \Delta\theta, \end{cases} \tag{1}$$

where (u,ω), p and θ denote the velocity field, pressure and departure of temperature from a linear profile while $\mathcal{P}$ and $\mathcal{R}$ denote Prandtl and Rayleigh numbers.

2 A Fixed Point Formulation

We will find the steady-state solutions of (1). We introduce the stream function Ψ, through the definition $(u,\omega) = (-\Psi_z, \Psi_x)$ so that $u_x + \omega_z = 0$. Cross-differentiating the equation of motion in (1) in order to eliminate the pressure p and setting $\Theta := \sqrt{\mathcal{P}\mathcal{R}}\theta$, we have

$$\begin{cases} \mathcal{P}\Delta^2\Psi &= \sqrt{\mathcal{P}\mathcal{R}}\,\Theta_x - \Psi_z\Delta\Psi_x + \Psi_x\Delta\Psi_z \quad \text{in} \quad \Omega, \\ -\Delta\Theta &= -\sqrt{\mathcal{P}\mathcal{R}}\,\Psi_x + \Psi_z\Theta_x - \Psi_x\Theta_z \quad \text{in} \quad \Omega, \\ \Psi &= 0, \quad \Delta\Psi = 0 \quad \text{on} \quad \partial\Omega, \\ \Theta(x,0) &= 0, \quad \Theta(x,\pi) = 0, \\ \Theta_x(0,z) &= 0, \quad \Theta_x(2\pi/a, z) = 0. \end{cases} \tag{2}$$

where we restrict the problem to the rectangular region $\{0 < x < 2\pi/a,\ 0 < z < \pi\}$ for given $a > 0$. We also impose periodic boundary conditions (period $2\pi/a$) in the horizontal direction and stress free boundary conditions ($u_z = 0$) on the surfaces $z = 0$, $z = \pi$ for the velocity field, and Dirichlet boundary conditions for the temperature field.

We can represent Ψ and Θ by the following double Fourier expansions because of the boundary condition:

$$\Psi = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin(amx)\sin(nz), \quad \Theta = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} B_{mn}\cos(amx)\sin(nz). \tag{3}$$

Setting $u := (\Psi, \Theta)$, $f_1(u) := \sqrt{\mathcal{PR}}\,\Theta_x - \Psi_z \Delta \Psi_x + \Psi_x \Delta \Psi_z$, $f_2(u) := -\sqrt{\mathcal{PR}}\,\Psi_x + \Psi_z \Theta_x - \Psi_x \Theta_z$ and $f(u) := (f_1(u), f_2(u))$, f is a bounded and continuous map from $H^3(\Omega) \times H^1(\Omega)$ to $L^2(\Omega) \times L^2(\Omega)$. Moreover, under the boundary conditions of (2), for all g_1, $g_2 \in L^2(\Omega)$, $\Delta^2 \bar{\Psi} = g_1$ and $-\Delta \bar{\Theta} = g_2$ have unique solutions $(\bar{\Psi}, \bar{\Theta}) \in H^4(\Omega) \times H^2(\Omega)$. When we denote the solutions by $\bar{\Psi} = (\Delta^2)^{-1} g_1$, $\bar{\Theta} = (-\Delta)^{-1} g_2$, an operator:

$$K := (\mathcal{P}^{-1}(\Delta^2)^{-1}, (-\Delta)^{-1}) : \; L^2(\Omega) \times L^2(\Omega) \to H^3(\Omega) \times H^1(\Omega)$$

is compact because of the compactness of the imbedding $H^4(\Omega) \hookrightarrow H^3(\Omega)$ and $H^2(\Omega) \hookrightarrow H^1(\Omega)$. Therefore, (2) is rewritten the fixed point equation:

$$u = Fu \tag{4}$$

for a compact operator $F := Kf$ on $H^3(\Omega) \times H^1(\Omega)$, then we can use Schauder's fixed point theorem.

3 Computable Verification Condition

For $\psi_{mn} := \sin(amx)\sin(nz)$, $\theta_{mn} := \cos(amx)\sin(nz)$, we define approximate subspaces $S_N^{(1)}$, $S_N^{(2)}$ by

$$S_N^{(1)} := \{\Psi_N \mid \Psi_N = \sum_{m=1}^{M_1} \sum_{n=1}^{N_1} \hat{A}_{mn} \psi_{mn}, \}, \qquad S_N^{(2)} := \{\Theta_N \mid \Theta_N = \sum_{m=0}^{M_2} \sum_{n=1}^{N_2} \hat{B}_{mn} \theta_{mn}\}.$$

Next, for given $g_1, g_2 \in L^2(\Omega)$, we define projections $P_N^{(1)} \bar{\Psi}$ and $P_N^{(2)} \bar{\Theta}$ of $\bar{\Psi} = (\Delta^2)^{-1} g_1 \in H^4(\Omega)$, $\quad \bar{\Theta} = (\Delta)^{-1} g_2 \in H^2(\Omega)$ by

$$\left\{ \begin{array}{rcll} \mathcal{P}(\Delta^2 P_N^{(1)} \bar{\Psi}, v_N^{(1)})_{L^2} & = & (g_1, v_N^{(1)})_{L^2} & \forall v_N^{(1)} \in S_N^{(1)}, \\ -(\Delta P_N^{(2)} \bar{\Theta}, v_N^{(2)})_{L^2} & = & (g_2, v_N^{(2)})_{L^2} & \forall v_N^{(2)} \in S_N^{(2)}, \end{array} \right.$$

where $(\cdot, \cdot)_{L^2}$ means the inner product on $L^2(\Omega)$. We can show that $P_N^{(1)} \bar{\Psi}$ and $P_N^{(2)} \bar{\Theta}_N$ coincide with (M_1, N_1)-truncation of $\bar{\Psi}$ and (M_2, N_2)-truncation of $\bar{\Theta}$ represented by the expansions in (3). Therefore, we can easily obtain the constructive a priori error estimates: $\|\bar{\Psi} - P_N^{(1)} \bar{\Psi}\|_{H^3}$, $\|\bar{\Theta} - P_N^{(2)} \bar{\Theta}\|_{H^1}$, and so on.

We define the projection P_N by $P_N := (P_N^{(1)}, P_N^{(2)})$, then the fixed point equation (4) can be decomposed as the finite dimensional part(projection) and infinite dimensional part(error) as follows:

$$\left\{ \begin{array}{rcl} P_N u & = & P_N F u, \\ (I - P_N) u & = & (I - P_N) F u, \end{array} \right.$$

and we can propose a computer algorithm to construct the set in $H^3(\Omega) \times H^1(\Omega)$ which satisfies the assumption of Schauder's fixed point theorem.