A Computer Assisted Proof of Bifurcated Solutions for the Heat Convection Problem

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Consider the Oberbeck-Boussinesq equations:

$$\begin{cases} u_t + uu_x + wu_z &= p_x + \mathcal{P}\Delta u, \\ w_t + uw_x + ww_z &= p_z - \mathcal{P}\mathcal{R}\,\theta + \mathcal{P}\Delta w, \\ u_x + w_z &= 0, \\ \theta_t + w + u\theta_x + w\theta_z &= \Delta\theta, \end{cases} \tag{1}$$

where $\mathcal{R}$ is the Rayleigh number and $\mathcal{P}$ is the Prandtl number.

It is well known that for small $\mathcal{R}$ the fluid conducts heat diffusively, and at a critial point $\mathcal{R}_\mathcal{C}$, heat is transposed through the fluid by convection. The origin of these rolls is the experiment by Bénard in 1900. He observed the establishment of a regular, steady pattern of flow cells in a thin horizontal layer of molten spermaceti with a free upper surface, then these cells which later came to be known as *Bénard cells*.

In 1916, Lord Rayleigh considered the linearized stability of (1) and found $\mathcal{R}_\mathcal{C}$ when both the upper and lower boundaries are taken to be stress-free. From the above-mentioned problem (1) (of course including three dimensional case) is called by *Rayleigh-Bénard convection*.

Although a large number of studies have been made on the Rayleigh-Bénard convection, theoretical results about the Rayleigh-Bénard convection are very few. It has been shown by Joseph that (1) has a unique trivial solution for $\mathcal{R} < \mathcal{R}_\mathcal{C}$. Iudovich and Rabinowitz proved that, for each $\mathcal{R}$ slightly exceeding the critical Rayleigh number $\mathcal{R}_\mathcal{C}$, the equation (1) has at least two nontrivial steady-state solutions. The stability analysis of the bifurcated solution in a small neighbourhood of the bifurcation points is considered by Kagei and Wahl. However, the global structure of bifurcated solutions after the critical Rayleigh point $\mathcal{R}_\mathcal{C}$ has not been known *theoretically* up to now.

In this talk, we propose an approach to prove the exsistence of the steady-state solutions for given $\mathcal{P}$ and $\mathcal{R}$ by a computer assisted proof, which enables us a tool for the study of the global bifurcation structure. This method is based on the infinite dimensional fixed-point theorem using Newton-like operator as well as on the spectral approximation and the constructive error estimates.