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A Numerical Method to Verify the Invertibility of Linear Elliptic Operators with Applications to Nonlinear Problems

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Abstract

In this paper, we propose a numerical method to verify the invertibility of second-order linear elliptic operators. By using the projection and the constructive *a priori* error estimates, the invertibility condition is formulated as a numerical inequality based upon the existing verification method originally developed by one of the authors. As a useful application of the result, we present a new verification method of solutions for nonlinear elliptic problems, which enables us to simplify the verification process. Several numerical examples that confirm the actual effectiveness of the method are presented.

AMS Subject Classifications: 35J25, 35J60, 65N25.

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1. Introduction

We consider the solvability of the linear elliptic boundary value problem of the form

$$\begin{aligned} \mathcal{L}u \equiv -\Delta u + b \cdot \nabla u + cu = g \;\; &\text{in} \quad \Omega, \\ u = 0 \;\; &\text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

that is equivalent to the invertibility of the operator $\mathcal{L}$ on a certain function space. Here, for $n = 1, 2, 3$, we assume that $b \in \left(W^1_\infty(\Omega)\right)^n$, $c \in L^\infty(\Omega)$, where $\Omega \subset R^n$ is a bounded convex domain with piecewise smooth boundary.

By using this result, we present a procedure to compute the operator norm corresponding to the inverse $\mathcal{L}^{-1}$, and then, we formulate a numerical verification method of solutions for the following nonlinear elliptic problems:

$$\begin{aligned} -\Delta u &= f(x, u, \nabla u) \quad & x \in \Omega, \\ u &= 0 & x \in \partial\Omega. \end{aligned} \tag{1.2}$$

Several works, based upon the principle originally found by one of the authors, have been presented for the numerical verification methods of solutions for (1.2), e.g., in [3], [6] etc. They use a method that consists of two procedures; one is a finite dimensional Newton-like iterative process, the other is the computation of the error caused by the gap between the finite and infinite dimension in each iterative procedure. In general, the method for the finite dimensional part utilizes a kind of interval Newton method; and it has been recently observed that in the case of having the term with a first order derivative ∇u, this iterative process sometimes fails due to the divergence of the interval computations. In order to overcome this difficulty, we considered an improvement, in [7], which adopts a technique that avoids directly solving the interval system of equations for the finite dimensional part.

In the present paper, we propose a new approach that utilizes the direct estimation of the norm of linearized inverse operators for (1.2) and yields further simplification of the verification procedures. This approach is in fact an extension of the method presented in [7]. Namely, we first verify the invertibility for linearlized operators and compute guaranteed norm bounds for its inverse by applying the same principle as for the existing method. Next, we show the existence of solutions for (1.2) by proving the contractivity of the Newton-like operator with a residual form. Another direct computational method of bounds for the linearized operator has already been proposed by Plum (see, e.g., [8], [10] etc.) using the eigenvalue enclosure methods with a homotopic technique. His method uses some homotopic steps with additional base functions and verified computations for relatively small matrix eigenvalue problems; this is considered a quite different approach from the present method. On the other hand, our verification procedure for nonlinear problems is very close to Plum's method based upon the infinite dimensional Newton's method of the residual type. Therefore, a comparison of these two methods, in respect to the total computational costs for verification of nonlinear problems, would very much depend on the individual problem.

In the below, we denote the L^2 inner product on Ω by $(\cdot, \cdot)$ and the norm by $\|\cdot\|_{L^2}$. And denote the usual L^2 Sobolev spaces on Ω by $H^k(\Omega)$ for any positive integer k. For the first-order Sobolev space $H_0^1(\Omega)$ with homogeneous boundary condition, we define the norm by $\|v\|_{H_0^1} := \|\nabla v\|_{L^2}$, and also define the H^2 semi-norm on Ω by, e.g., when $n = 2$,

$$|u|_{H^2} = \left(\|u_{xx}\|_{L^2}^2 + 2\left\|u_{xy}\right\|_{L^2}^2 + \left\|u_{yy}\right\|_{L^2}^2\right)^{\frac{1}{2}}.$$

For $n = 1$ or $n = 3$, analogously defined.

2. Invertibility Condition of Linear Elliptic Operators

In the present section, we consider the numerical verification condition of invertibility for the operator $\mathcal{L}$ defined by (1.1), as well as we present a method to estimate the norm of the inverse operator corresponding to $\mathcal{L}^{-1}$.

We now introduce the finite dimensional subspace S_h of $H_0^1(\Omega)$ depending on the parameter h with nodal functions $\{\phi_i\}_{1\le i\le N}$. And, for each $v \in H_0^1(\Omega)$, define the H_0^1-projection $P_h v \in S_h$ by

$$(\nabla(v - P_h v), \nabla\phi_h) = 0, \qquad \forall\phi_h \in S_h.$$

Further, we assume that there exists a positive constant C_0 which can be numerically estimated satisfying, for any $u \in H^2(\Omega) \cap H_0^1(\Omega)$,

$$\|u - P_h u\|_{H_0^1} \le C_0 h |u|_{H^2}. \tag{2.1}$$

Notice that the invertibility of the elliptic operator $\mathcal{L}$ defined in (1.1) is equivalent to the unique solvability of the fixed point equation

$$u = Au, \tag{2.2}$$

where the compact operator $A : H_0^1 \longrightarrow H_0^1$ is defined by $Au := \Delta^{-1}(b \cdot \nabla u + cu)$ and where Δ^{-1} stands for the solution operator of the Poisson equation with homogeneous boundary condition.

Now, according to the usual verification principle, e.g., [6], we formulate a sufficient condition for which Eq. (1.2) has a unique solution. As the preliminary, we define the matrices $\mathbf{G} = (\mathbf{G}_{i,j})$ and $\mathbf{D} = (\mathbf{D}_{i,j})$ by

$$\begin{aligned} \mathbf{G}_{i,j} &= (\nabla\phi_j, \nabla\phi_i) + (b \cdot \nabla\phi_j, \phi_i) + (c\phi_j, \phi_i), \\ \mathbf{D}_{i,j} &= (\nabla\phi_j, \nabla\phi_i), \qquad \text{for} \quad 1 \le i, j \le N. \end{aligned}$$

Let $\mathbf{L}$ be a lower triangular matrix satisfying the Cholesky decomposition: $\mathbf{D} = \mathbf{L}\mathbf{L}^T$. And we denote the matrix norm by $\|\cdot\|_E$ induced from the Euclidean norm $|\cdot|_E$ in R^N. Also, we define the following constants:

$$\begin{aligned} &C_b = \|\, |b|_E \,\|_{L^\infty}, \quad C_b' = \|\nabla \cdot b\|_{L^\infty}, \quad C_c = \|c\|_{L^\infty}, \\ &C_1 = C_0(\|\nabla \cdot b\|_{L^\infty} C_p + C_b), \qquad C_3 = C_b + C_c C_p, \\ &C_2 = C_0 C_c C_p, \qquad\qquad\qquad\qquad C_4 = C_b + C_0 C_c h, \end{aligned}$$

where $\|\cdot\|_{L^\infty}$ means L^∞ norm on Ω and C_p is a Poincaré constant such that $\|\phi\|_{L^2} \le C_p \|\phi\|_{H_0^1}$ for arbitrary $\phi \in H_0^1(\Omega)$. Then we have the following main result of this paper.

Theorem 2.1: *If the matrix* $\mathbf{G}$ *is invertible and, for the constants defined above,*

$$C_0 h (C_3 M (C_1 + C_2) h + C_4) < 1$$

holds, then the operator $\mathcal{L}$ *defined in (1.1) is invertible. Here,* $M \equiv \|\mathbf{L}^T \mathbf{G}^{-1} \mathbf{L}\|_E$ *and* C_0 *is the same constant as in (2.1).*

Remark 1: *The main cost for checking the invertibility condition consists of the guaranteed estimation of* $\|\mathbf{L}^T \mathbf{G}^{-1} \mathbf{L}\|_E$*. First, we compute the matrix* $\mathbf{L}$ *by the interval Cholesky-decomposition. Next, by using the approximate LU decompositin of* $\mathbf{G}$ *and some error estimates, we enclose the guranteed inverse* $\mathbf{G}^{-1}$*. Finally, we make a verified computation of the largest singular value for the matrix* $\mathbf{L}^T \mathbf{G}^{-1} \mathbf{L}$*, which is equal to the square root of the largest eigenvalue of symmetric matrix* $\mathbf{L^T G^{-T} D G^{-1} L}$*, to obtain the desired estimation.*

Proof: First, as usual, we decompose the equation $u = Au$ as

$$P_h u = P_h A u,$$
$$(I - P_h)u = (I - P_h)Au,$$

where I implies the identity map on $H_0^1(\Omega)$.

Next, according to the same formulation to that in [4], [6] etc., we define two operators by

$$N_h u \equiv P_h u - [I - A]_h^{-1} P_h (I - A) u$$

and

$$Tu \equiv N_h u + (I - P_h)Au,$$

respectively, where $[I - A]_h^{-1}$ means the inverse of $P_h(I - A)|_{S_h} : S_h \longrightarrow S_h$. Note that if we define the Galerkin approximation A_h on S_h of the operator A, then $[I - A]_h^{-1}$ coincides with $(I - A_h)^{-1}$ on S_h. The existence of the operator $[I - A]_h^{-1}$ is assumed, which is equivalent to the regularity of the corresponding matrix, and is numerically followed by the unique solvability of the linear system of equations in the verification process.

We now, for positive real numbers α and γ, define the set $U = U_h + U_\perp$ by

$$U_h := \left\{ u_h \in S_h : \|u_h\|_{H_0^1} \le \gamma \right\},$$
$$U_\perp := \left\{ u_\perp \in S_h^\perp : \|u_\perp\|_{H_0^1} \le \alpha \right\},$$

where $S_h^\perp$ stands for the orthogonal complement of S_h in $H_0^1(\Omega)$. Then, by the fact that $u = Au$ is equivalent to $u = Tu$, in order to prove the unique existence of a solution to (2.2) in the set U, it suffices to show the inclusion $TU \overset{\circ}{\subset} U$ due to the linearity of the equation (e.g., [12]), where $TU \overset{\circ}{\subset} U$ implies $\overline{TU} \subset \overset{\circ}{U}$, i.e., the closure of TU is included by the interior of U.

Further notice that a sufficient condition of this inclusion can be written as

$$\|N_h U\|_{H_0^1} \equiv \sup_{u \in U} \|N_h u\|_{H_0^1} < \gamma, \tag{2.3}$$

$$\begin{aligned} \|(I - P_h)AU\|_{H_0^1} &\equiv \sup_{u \in U} \|(I - P_h)Au\|_{H_0^1} \\ &\le C_0 h \sup_{u \in U} |Au|_{H^2} \\ &\le C_0 h \sup_{u \in U} \|b \cdot \nabla u + cu\|_{L^2} < \alpha, \end{aligned} \tag{2.4}$$

where we have used the estimate (2.1) and well known inequality $|\phi|_{H^2} \le \|\Delta\phi\|_{L^2}$ for $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$ on the convex domain Ω.

In the below, we estimate $\|N_h u\|_{H_0^1}$ and $\|b\cdot\nabla u+cu\|_{L^2}$ in (2.3) and (2.4), respectively. First, for arbitrary $u = u_h + u_\perp \in U_h + U_\perp$, setting $\psi_h := N_h(u_h + u_\perp)$, we have

$$\begin{aligned}\psi_h &= u_h - [I-A]_h^{-1} P_h (I-A)(u_h+u_\perp)\\ &= [I-A]_h^{-1} P_h A u_\perp.\end{aligned} \tag{2.5}$$

Now, note that for $v_h := P_h A u_\perp \in S_h$ we have

$$(\nabla\psi_h, \nabla\phi_h) + (b\cdot\nabla\psi_h, \phi_h) + (c\psi_h, \phi_h) = (\nabla v_h, \nabla\phi_h), \qquad \forall \phi_h \in S_h. \tag{2.6}$$

Denoting

$$\psi_h := \sum_{i=1}^N w_i\phi_i \qquad \text{and} \qquad v_h := \sum_{i=1}^N v_i\phi_i, \tag{2.7}$$

from (2.6) we have a matrix equation of the form

$$\mathbf{G}\vec{w} = \mathbf{D}\vec{v}. \tag{2.8}$$

Here, $\vec{w} = (w_1, w_2, \ldots, w_N)^T$ and $\vec{v} = (v_1, v_2, \ldots, v_N)^T$ are coefficient vectors of ψ_h and v_h, respectively. Therefore, from (2.7) and (2.8), it follows that

$$\begin{aligned}\|\psi_h\|_{H_0^1}^2 &= \vec{w}^T\mathbf{D}\vec{w}\\ &= \vec{w}^T\mathbf{D}\mathbf{G}^{-1}\mathbf{D}\vec{v}\\ &= (\mathbf{L}^T\vec{w})^T(\mathbf{L}^T\mathbf{G}^{-1}\mathbf{L})(\mathbf{L}^T\vec{v})\\ &\le \|\mathbf{L}^T\vec{w}\|_E\|\mathbf{L}^T\mathbf{G}^{-1}\mathbf{L}\|_E\|\mathbf{L}^T\vec{v}\|_E\\ &= \|\psi_h\|_{H_0^1}\|\mathbf{L}^T\mathbf{G}^{-1}\mathbf{L}\|_E\|v_h\|_{H_0^1}.\end{aligned}$$

Note that, from the above fact, we have $\|\mathbf{L}^T\mathbf{G}^{-1}\mathbf{L}\|_E = \|[I-A]_h^{-1}\|_{H_0^1}$. Thus, defining $M \equiv \|\mathbf{L}^T\mathbf{G}^{-1}\mathbf{L}\|_E$, we obtain

$$\begin{aligned}\|\psi_h\|_{H_0^1} &\le M\|P_h A u_\perp\|_{H_0^1}\\ &= M\|P_h\Delta^{-1}(b\cdot\nabla u_\perp + cu_\perp)\|_{H_0^1}\\ &\le M\|\Delta^{-1}(b\cdot\nabla u_\perp + cu_\perp)\|_{H_0^1}.\end{aligned} \tag{2.9}$$

Next, letting $\psi_1 := \Delta^{-1}(b\cdot\nabla u_\perp)$, some simple calculations yields that

$$\begin{aligned}\|\psi_1\|_{H_0^1}^2 = (\nabla\psi_1, \nabla\psi_1) &= (-\Delta\psi_1, \psi_1)\\ &= (-b\cdot\nabla u_\perp, \psi_1)\\ &\le \|u_\perp\|_{L^2}\|\mathrm{div}(b\psi_1)\|_{L^2}\\ &\le C_0 h\alpha(\|\nabla\cdot b\|_{L^\infty}C_p + C_b)\|\psi_1\|_{H_0^1},\end{aligned} \tag{2.10}$$

where we have used the fact $\|u_\perp\|_{L^2} \le C_0 h\alpha$. Furthermore, setting $\psi_2 := \Delta^{-1}(cu_\perp)$ and by applying the similar argument to the above, we have

$$\|\psi_2\|_{H_0^1}^2 \le C_c C_p C_0 h\alpha\|\psi_2\|_{H_0^1}. \tag{2.11}$$

Thus, by Eqs. (2.9)–(2.11), we obtain the following estimate for the finite dimensional part:

$$\|N_h U\|_{H_0^1} \leq M(C_1 + C_2)h\alpha, \tag{2.12}$$

where $C_1 \equiv C_0(\|\nabla \cdot b\|_{L^\infty} C_p + C_b)$, $C_2 \equiv C_c C_p C_0$.

Next, observe that

$$\begin{aligned}
\|b \cdot \nabla u_h + cu_h\|_{L^2} &\leq C_b \|u_h\|_{H_0^1} + C_c C_p \|u_h\|_{H_0^1} \\
&\leq (C_b + C_c C_p)\gamma, \\
\|b \cdot \nabla u_\perp + cu_\perp\|_{L^2} &\leq C_b \|u_\perp\|_{H_0^1} + C_c \|u_\perp\|_{L^2} \\
&\leq (C_b + C_0 C_c h)\alpha.
\end{aligned}$$

Therefore, by using (2.4) and the triangle inequality, we have

$$\|(I - P_h)AU\|_{H_0^1} \leq C_0 h(C_3\gamma + C_4\alpha), \tag{2.13}$$

where $C_3 \equiv C_b + C_c C_p$, $C_4 \equiv C_b + C_0 C_c h$.

Now from Eqs. (2.12) and (2.13), the invertibility conditions (2.3) and (2.4) are reduced to

$$M(C_1 + C_2)h\alpha < \gamma, \tag{2.14}$$

$$C_0 h(C_3\gamma + C_4\alpha) < \alpha. \tag{2.15}$$

For arbitrary small $\varepsilon > 0$, if we set $\gamma := M(C_1 + C_2)h\alpha + \varepsilon$, then the condition (2.14) clearly holds. Therefore, by substituting it for (2.15) we have

$$C_0 h(C_3(M(C_1 + C_2)h\alpha + \varepsilon) + C_4\alpha) < \alpha,$$

which is equivalent to

$$1 - C_0 h(C_3 M(C_1 + C_2)h + C_4) > 0.$$

Thus the desired conclusion is obtained. □

Remark 2: *The conditions (2.3) and (2.4) are equivalent to* $|||T||| < 1$ *in some scaled norm* $|||\cdot|||$ *in* H_0^1, *e.g.,* $|||v|||^2 \equiv ||P_h v||^2_{H_0^1}/\gamma^2 + ||(I - P_h)v||^2_{H_0^1}/\alpha^2$. *Then, the invertibility of theoperator* $I - T$ *follows by the convergence of the Neumann series.*

When the coefficient function b of the first-order term is not differentiable, we have the following alternative condition.

Corollary 1: *For the operator* $\mathcal{L}$ *defined in 1.1, let* $b \in (L^\infty(\Omega))^n$. *If*

$$C_0 h(C_3 M(\hat{C}_1 + C_2 h) + C_4) < 1,$$

then the operator $\mathcal{L}$ *defined in (1.1) is invertible. Here,* $\hat{C}_1 = \sqrt{n} C_b C_p$.

Proof: The difference from the proof of Theorem 2.1 is only the part concerning the estimates (2.10). Corresponding estimates are now

$$\begin{aligned}\|\psi_1\|_{H_0^1}^2 = (-\Delta\psi_1, \psi_1) = (-b \cdot \nabla u_\perp, \psi_1) &\\ &\le C_b \|u_\perp\|_{H_0^1} \|\psi_1\|_{L^2} \\ &\le C_b C_p \alpha \|\psi_1\|_{H_0^1},\end{aligned}$$

which proves the corollary. □

Now our next purpose is the estimation of the operator norm $\|(I - A)^{-1}\|_{H_0^1}$ corresponding to the norm for $\mathcal{L}^{-1} : H^{-1} \to H_0^1$.

Theorem 2.2: *Under the same assumptions in Theorem 2.1, provided that*

$$\kappa \equiv C_0 h(C_3 M(C_1 + C_2)h + C_4) < 1,$$

then the following estimation holds:

$$\|(I - A)^{-1}\|_{H_0^1} \le \|R + S\|_E^{\frac{1}{2}} =: \mathcal{M}, \tag{2.16}$$

where the 2×2 *matrices* R, S *are defined by*

$$S = \begin{bmatrix} s_h^2 & s_h s_\perp h \\ s_h s_\perp h & s_\perp^2 h^2 \end{bmatrix}, \quad R = \begin{bmatrix} r_h^2 h^2 & r_h r_\perp h \\ r_h r_\perp h & r_\perp^2 \end{bmatrix}.$$

Here, $(s_h, s_\perp)$, $(r_h, r_\perp)$ *are given as follows:*

$$\begin{aligned} s_h &= M[r_h(C_1 + C_2)h^2 + 1], & s_\perp &= M r_\perp (C_1 + C_2), \\ r_h &= C_0 C_3 M r_\perp, & r_\perp &= 1/(1 - \kappa). \end{aligned}$$

Proof: Let ψ be an arbitrary element in $H_0^1(\Omega)$. Then, by the Fredholm alternative theorem, the invertibility of $(I - A)$ implies that there exists a unique element $u \in H_0^1(\Omega)$ satisfying $(I - A)u = \psi$. When we set

$$\begin{aligned} N_h u &:= P_h u - [I - A]_h^{-1} P_h((I - A)u - \psi), \\ T u &:= N_h u + (I - P_h)(Au + \psi), \end{aligned}$$

it is readily seen that $(I - A)u = \psi$ is equivalent to $Tu = u$. Using the unique decompositions $u = u_h + u_\perp$ and $\psi = \psi_h + \psi_\perp$ in $H_0^1(\Omega) = S_h \oplus S_h^\perp$, by some simple calculations, we have

$$\begin{aligned} u_h &= [I - A]_h^{-1}(P_h A u_\perp + P_h \psi), \\ u_\perp &= (I - P_h)A(u_h + u_\perp) + (I - P_h)\psi. \end{aligned} \tag{2.17}$$

Hence, taking notice of $M = \|[I - A]_h^{-1}\|_{H_0^1}$ and the estimates in the proof of Theorem 2.1, we have

$$\begin{aligned} \|u_h\|_{H_0^1} &\le M\|P_h A u_\perp + P_h\psi\|_{H_0^1} \\ &\le M(C_1 + C_2)h\|u_\perp\|_{H_0^1} + M\|P_h\psi\|_{H_0^1}, \end{aligned} \tag{2.18}$$

$$\begin{aligned} \|u_\perp\|_{H_0^1} &\le \|(I - P_h)A(u_h + u_\perp)\|_{H_0^1} + \|(I - P_h)\psi\|_{H_0^1} \\ &\le C_0 h(C_3\|u_h\|_{H_0^1} + C_4\|u_\perp\|_{H_0^1}) + \|(I - P_h)\psi\|_{H_0^1}. \end{aligned} \tag{2.19}$$

Substituting the estimate of $\|u_h\|_{H_0^1}$ in (2.18) for the last right-hand side of (2.19) and solving it with respect to $\|u_\perp\|_{H_0^1}$, we get

$$\begin{aligned} \|u_\perp\|_{H_0^1} &\le (C_0 C_3 h M)/(1 - \kappa)\|P_h\psi\|_{H_0^1} + 1/(1 - \kappa)\|(I - P_h)\psi\|_{H_0^1} \\ &= r_h h\|P_h\psi\|_{H_0^1} + r_\perp\|(I - P_h)\psi\|_{H_0^1}. \end{aligned} \tag{2.20}$$

Thus we also have by (2.18)

$$\begin{aligned} \|u_h\|_{H_0^1} &\le M(C_1 + C_2)h\left(r_h h\|P_h\psi\|_{H_0^1} + r_\perp\|(I - P_h)\psi\|_{H_0^1}\right) + M\|P_h\psi\|_{H_0^1} \\ &\le M\left[r_h(C_1 + C_2)h^2 + 1\right]\|P_h\psi\|_{H_0^1} + M r_\perp(C_1 + C_2)h\|(I - P_h)\psi\|_{H_0^1} \\ &= s_h\|P_h\psi\|_{H_0^1} + s_\perp h\|(I - P_h)\psi\|_{H_0^1}. \end{aligned} \tag{2.21}$$

Therefore, we obtain the desired conclusion from (2.20) and (2.21). □

Moreover, we have the following estimates corresponding to Corollary 1.

Corollary 2: *Under the same assumption as in Corollary 1, if*

$$\hat{\kappa} \equiv C_0 h(C_3 M(\hat{C}_1 + C_2 h) + C_4) < 1,$$

then

$$\|(I - A)^{-1}\|_{H_0^1} \le (\hat{R}^2 + \hat{S}^2)^{\frac{1}{2}} =: \hat{\mathcal{M}}. \tag{2.22}$$

Here, $\hat{R}$ and $\hat{S}$ are defined as

$$\hat{R} := (C_0 C_3 h M + 1)/(1 - \hat{\kappa}) \quad \textit{and} \quad \hat{S} := \{(\hat{C}_1 + C_2 h)R + 1\}M.$$

We now note that the following *a priori* estimate of the solution to (1.1) is obtained.

Theorem 2.3:

$$\|u\|_{H_0^1} \le \|(I - A)^{-1}\|_{H_0^1}\|g\|_{H^{-1}}.$$

Particularly,

$$\|u\|_{H_0^1} \le C_p\|(I - A)^{-1}\|_{H_0^1}\|g\|_{L^2} \quad \textit{for } g \in L^2(\Omega).$$

Indeed, defining $\psi := -\Delta^{-1}g$, then taking account that $(I-A)u = \psi$ and that

$$\|\psi\|_{H_0^1}^2 = (\nabla\psi, \nabla\psi)\langle -\Delta\psi, \psi\rangle = \langle g, \psi\rangle \le \|g\|_{H^{-1}}\|\psi\|_{H_0^1},$$

where $\langle\cdot,\cdot\rangle$ stands for the duality pairing in $H_0^1(\Omega)$. The second part follows from the Poincaré inequality.

3. Applications

In this section, we mention about the actual applications of the results obtained in the previous section to the verification of solutions for nonlinear elliptic problem (1.2). We assume that the nonlinear map $f(u) \equiv f(\cdot, u, \nabla u)$ from $H_0^1(\Omega)$ into $L^2(\Omega)$ is continuous and bounded.

3.1. Preliminary

In this subsection, we transform the original boundary value problem (1.2) into the so-called residual equation by using an approximate solution $\hat{u}_h \in S_h \subset H_0^1(\Omega)$ defined by

$$(\nabla\hat{u}_h, \nabla\phi_h) = (f(\hat{u}_h), \phi_h), \qquad \forall\phi_h \in S_h. \tag{3.1}$$

For the effective computation of the solution for (3.1) with guaranteed accuracy, refer, for example, [1], [11] etc.

Next, we define the $\bar{u} \in H_0^1(\Omega) \cap H^2(\Omega)$ by the solution of Poisson's equation

$$\begin{aligned} -\Delta\bar{u} &= f(\hat{u}_h) \ \text{ in } \ \Omega, \\ \bar{u} &= 0 \qquad\ \ \text{on } \partial\Omega. \end{aligned} \tag{3.2}$$

Further, let define residues by

$$u - \hat{u}_h = (u - \bar{u}) + (\bar{u} - \hat{u}_h), \qquad w := u - \bar{u}, \quad v_0 := \bar{u} - \hat{u}_h. \tag{3.3}$$

Note that v_0 is an unknown function but its norm can be computed by an *a priori* and *a posteriori* techniques (e.g., see [5], [13]). Thus, using the residues in (3.3), concerned problem is reduced to the following residual form :

$$\begin{aligned} -\Delta w &= f(w + v_0 + \hat{u}_h) - f(\hat{u}_h) \ \text{ in } \ \Omega, \\ w &= 0 \qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega. \end{aligned} \tag{3.4}$$

Hence, denoting the Fréchet derivative at $\hat{u}_h$ by $f'(\hat{u}_h)$, the Newton-type residual equation for (3.4) is written as

$$\begin{aligned} -\Delta w - f'(\hat{u}_h)w &= g_r(w) \ \text{ in } \ \Omega, \\ w &= 0 \qquad\ \ \text{on } \partial\Omega, \end{aligned} \tag{3.5}$$

where $g_r(w) \equiv f(w + v_0 + \hat{u}_h) - f(\hat{u}_h) - f'(\hat{u}_h)w$.

In the above, we assumed that the approximate solution $\hat{u}_h$ is defined as an element in $H_0^1(\Omega)$, i.e., C^0-element. When we use the function satisfying $\hat{u}_h \in H^2(\Omega)$, i.e., C^1-element, we can get more simpler residual Newton-type equation without v_0 of the form

$$\begin{aligned} -\Delta w - f'(\hat{u}_h)w &= g_d(w) \quad \text{in} \quad \Omega, \\ w &= 0 \qquad\quad \text{on } \partial\Omega, \end{aligned} \tag{3.6}$$

where $w := u - \hat{u}_h$ and $g_d(w) := f(w + \hat{u}_h) + \Delta\hat{u}_h - f'(\hat{u}_h)w$. For another type of simple residual formulation for C^0-element, refer [4] or [10] in which some H^{-1} arguments are effectively used.

3.2. *Verification Conditions*

We now write down again the nonlinear boundary value problem of the form:

$$\begin{aligned} \mathcal{L}w \equiv -\Delta w - f'(\hat{u}_h)w &= g(w) \quad \text{in} \quad \Omega, \\ w &= 0 \qquad\quad \text{on } \partial\Omega, \end{aligned} \tag{3.7}$$

where $g(w) \equiv g_r(w)$ or $g(w) \equiv g_d(w)$. If $\mathcal{L}$ is invertible, then (3.7) is rewritten as the fixed point form

$$w = F(w) \left(\equiv \mathcal{L}^{-1}g(w)\right). \tag{3.8}$$

Notice that the Newton-like operator F in (3.8) is compact on $H_0^1(\Omega)$ from the assumptions on f, and that it is expected to be a contraction map on some neighborhood of zero.

Therefore, we consider the set, which we often refer as the *candidate set*, of the form $W_\alpha \equiv \{w \in H_0^1(\Omega) : \|w\|_{H_0^1} \le \alpha\}$.

First, for the existential condition of solutions, we need to choose the set W_α, which is equivalent to determine a positive number α, satisfying the following criterion based on the Schauder fixed point theorem:

$$F(W_\alpha) \subset W_\alpha. \tag{3.9}$$

And next, for the proof of local uniqueness within W_α, the following contraction property is needed on the same set W_α in (3.9):

$$\|F(w_1) - F(w_2)\|_{H_0^1} \le k\|w_1 - w_2\|_{H_0^1}, \quad \forall w_1, w_2 \in W_\alpha, \tag{3.10}$$

for some constant $0 < k < 1$. Notice that, in the above case, the Schauder fixed point theorem can be replaced by the Banach fixed point theorem, which might yields an advantage if we apply our method to noncompact problems.

For (3.9), from Theorem 2.3, a sufficient condition can be written as

$$\|F(W_\alpha)\|_{H_0^1} \equiv \sup_{w \in W_\alpha} \|F(w)\|_{H_0^1} \le \mathcal{M}_1 \sup_{w \in W_\alpha} \|g(w)\|_{L^2} \le \alpha, \tag{3.11}$$

where $\mathcal{M}_1 \equiv C_p \mathcal{M}$, and $\mathcal{M}$ is the norm of the operator $\mathcal{L}^{-1} : H^{-1} \to H_0^1$ defined in Theorem 2.2.

On the other hand, for the verification of local uniqueness condition (3.10) on W_α, in general, we use the following deformation:

$$g(w_1) - g(w_2) = \Phi(w_1, w_2)(w_1 - w_2),$$

where $\Phi(w_1, w_2)$ denotes a function in w_1 and w_2, for example, if $g(w) = w^2$, then $\Phi(w_1, w_2) = w_1 + w_2$. Therefore, Condition (3.10) reduces to find a constant $0 < k < 1$ satisfying the inequality of the form

$$\mathcal{M}_1 \|\Phi(w_1, w_2)(w_1 - w_2)\|_{L^2} \le k\|w_1 - w_2\|_{H_0^1}, \quad \forall w_1, w_2 \in W_\alpha. \tag{3.12}$$

4. Numerical Examples

Example 4.1 (Emden's equation):

$$\begin{aligned} -\Delta u &= u^2 \;\text{ in }\; \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{4.1}$$

In this case, $\mathcal{L}$ and $g(w)$ in (3.7) are given as follows:

$$\begin{aligned} \mathcal{L}w &\equiv -\Delta w - 2\hat{u}_h w, \\ g_r(w) &\equiv w^2 + 2v_0 w + v_0^2 + 2\hat{u}_h v_0. \end{aligned} \tag{4.2}$$

Therefore, for the candidate set $W_\alpha = \{w \in H_0^1(\Omega) : \|w\|_{H_0^1} \le \alpha\}$, Condition (3.11) is given by

$$\mathcal{M}_1 \sup_{w \in W_\alpha} \|w^2 + 2v_0 w + v_0^2 + 2\hat{u}_h v_0\|_{L^2} \;\le\; \alpha. \tag{4.3}$$

By (4.3) and some calculations using the several kinds of norms, e.g., [4], [13] etc., we obtain the existential condition (3.11) of the form:

$$\mathcal{M}_1(K_2\alpha^2 + K_1\alpha + K_0) \le \alpha, \tag{4.4}$$

where $K_i,\ 0 \le i \le 2$, are constants dependent on the norms of $\hat{u}_h$ and v_0. It implies that, for any positive number α satisfying the quadratic inequality (4.4), there exists at least one solution in the set of the form $\hat{u}_h + v_0 + W_\alpha$. Note that such an α exists if and only if $\mathcal{M}_1(K_1 + 2\sqrt{K_0 K_1}) \le 1$. Also, notice that a sufficient condition corresponding to the relation (3.12) can be similarly and readily treated, and it leads to a simple linear inequality in α such that $\mathcal{M}_1(2K_2\alpha + K_1) < 1$. Thus, we can determine two bounds for α, i.e., α_E and α_U, for which we assure the existence and the uniqueness of solutions, respectively. Table 1 shows the computational results for the domain $\Omega = (0, 1) \times (0, 1)$ using piecewise quadratic C^0 finite element space S_h with several mesh sizes. Then the contant C_0 in (2.1) can be taken as $1/2\pi$ ([5]). In Table 1, "smallest α_E" and "largest α_U" indicate the smallest and the largest bounds α satisfying the verification conditions (3.11) and (3.12), respectively.

Table 1. Verification results for Example 4.1

$1/h$	$\mathcal{M}_1$	K_2	K_1	K_0	smallest α_E	largest α_U
5	1.5752	$1/\pi^2$	0.2441	4.0418	fail	1.9278363
10	0.7521	$1/\pi^2$	0.0483	0.5195	0.4194762	6.3223191
20	0.6485	$1/\pi^2$	0.0088	0.0635	0.0415689	7.5651910
$1/h$	M	C_1	C_2	C_3	C_4	$\|v_0\|_{H_0^1}$
5	2.7265	0	2.1124	13.2729	1.8770	2.0748884
10	2.7455	0	2.1103	13.2594	0.9375	0.5480243
20	2.7467	0	2.1025	13.2106	0.4670	0.1356515

Example 4.2 (Burgers equation):

$$\begin{aligned} \Delta u &= \lambda(u\cdot\nabla)u \quad \text{in} \quad \Omega, \\ u &= \varphi(x,y) \qquad \text{on } \partial\Omega, \end{aligned} \tag{4.5}$$

where λ is a parameter and $\varphi(x,y) \equiv \frac{1}{2}xy(1-y)$.

In this case, we consider a modified candidate set of the form

$$W_\alpha \equiv \left\{ w \in H_0^1(\Omega)\cap L^\infty(\Omega) : \max\{\|\nabla w\|_{L^2}, \|w\|_\infty\} \le \alpha \right\}. \tag{4.6}$$

Namely, we enclose the solution of (4.5) in the Banach space $X \equiv H_0^1(\Omega)\cap L^\infty(\Omega)$ with norm $\|w\|_X \equiv \max\{\|\nabla w\|_{L^2}, \|w\|_\infty\}$. Further we need the inverse norm estimates in the following L^∞ sense:

$$\|v\|_{L^\infty} \le \mathcal{M}_\infty \|\mathcal{L}v\|_{L^2}, \quad \forall v \in H^2(\Omega)\cap H_0^1(\Omega),$$

where $\mathcal{M}_\infty$ can be computed by using $\mathcal{M}_1$ in Sect. 2 and the constructive approach to the imbedding theory described in [8], [9].

Thus, the condition for existence is written as

$$\max(\mathcal{M}_1, \mathcal{M}_\infty) \sup_{w\in W_\alpha} \|g(w)\|_{L^2} \le \alpha. \tag{4.7}$$

Then, the linearized operator $\mathcal{L}$ and the right-hand side $g(w)$ of (3.7) are as follows:

$$\begin{aligned} \mathcal{L}w &\equiv -\Delta w + (\hat{u}_h\cdot\nabla)w + (w\cdot\nabla)\hat{u}_h, \\ g_r(w) &\equiv -\lambda\left[((w+v_0)\cdot\nabla)(w+v_0) + (\hat{u}_h\cdot\nabla)v_0 + (v_0\cdot\nabla)\hat{u}_h\right]. \end{aligned} \tag{4.8}$$

The verification conditions using α are similarly represented as in the previous example. That is, corresponding to the condition (4.4), it also leads to the inequality in α of the quadratic form such that $c_2\alpha^2 + c_1\alpha + c_0 \le \alpha$, where $c_i,\ 0 \le i \le 2$, are constants determined similarly as K_i in the previous example. Particularly, for the efficient computations, we used the L^∞ residual method for v_0 ([5]). And the uniqueness condition is also similarly given as before. The verification results for the parameter $\lambda = 10$ are shown in Table 2 with the same domain Ω and approximation subspace S_h as before.

Table 2. Verification results for Example 4.2 for $\lambda = 10$

$1/h$	$\mathcal{M}_1$	$\mathcal{M}_\infty$	c_2	c_1	c_0	smallest α_E	largest α_U
5	0.2674	0.6373	$10\sqrt{2}$	0.2253	0.0100	0.0081600	0.0475085
10	0.2444	0.5981	$10\sqrt{2}$	0.0596	0.0023	0.0015055	0.0569962
20	0.2344	0.5811	$10\sqrt{2}$	0.0144	0.0005	0.0003238	0.0603275
$1/h$	M	C_1	C_2	C_3	C_4	$\|v_0\|_{H_0^1}$	$\|v_0\|_\infty$
5	1.0029	0.5013	0.2194	3.1499	1.9662	0.0047170	0.0112155
10	1.0030	0.5030	0.2211	3.1609	1.8694	0.0012529	0.0029667
20	1.0030	0.5036	0.2220	3.1644	1.8187	0.0003013	0.0007184

Remark 3: *The computational efficiency of the above results, in Example 4.1, was almost similar to that the existing methods up to now, e.g., comparing with [13]. But, the determination of the range for existence and/or uniqueness as shown in the tables might be impossible for those methods up to now. Particularly, we can find rather wide range which contains no solutions. For example, from Tables 1 and 2, we can conclude that there are no solutions at all for α in* $[0.04157, 7.56519]$ *and in* $[0.0003238, 0.06032]$, *respectively. This property should be useful and powerful for the purpose to prove the nonexistence theorem in various kinds of problems.*

Remark 4: *For the present cases, we separately verified the existence and uniqueness by the criteria (3.9) and (3.10), respectively. We can also use another method to prove them simultaneously. Namely, the condition*

$$F(0) + F'(W)W \overset{\circ}{\subset} W$$

is satisfied for the candidate set W, then it implies that a locally unique solution is enclosed in W ([14]).

Remark 5: *All computations in Tables 1 and 2 are carried out on a Dell Latitude C400 Intel Pentium Mobile CPU 866MHz by using INTLAB 4.1.2, a tool box in MATLAB 6.5.1 developed by Rump [11] for self-validating algorithms. Therefore, all numerical values in these tables are verified data in the sense of strictly rounding error control.*

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