VERIFIED COMPUTATIONS OF SOLUTIONS FOR NONDIFFERENTIABLE ELLIPTIC EQUATIONS RELATED TO MHD EQUILIBRIA

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1 Introduction

We consider a numerical technique to enclose the exact solution with guaranteed error bounds for nondifferentiable nonlinear elliptic equations of second order. Our method is a kind of Newton-like method using an enclosing technique combined with the explicit error estimate for finite element approximations. In [1], we proposed a verification procedure which improves the original one in [2], and we presented some numerical examples to confirm the better convergence property, for example, the verified solutions for an equation appeared in the mathematical biology.

In this paper, we extend the verification procedure presented by [1] to nondifferentiable nonlinear elliptic problem related to MHD equilibria, and we construct a computing algorithm which automatically encloses the solution with guaranteed error bounds.

In the following section, we formulate a numerical verification method for a parametrized nonlinear nondifferentiable elliptic problem. This formulation is based on the infinite dimensional fixed point theorem using the Newton-like operator and the error estimates for finite element approximations. In Section 3, we introduce two concepts, rounding and rounding error, to deal with the infinite dimensional operator in a computer. The rounding is computed by solving the linear system of equation with interval righthand side. As for the rounding error mainly consists of the bounding of arithmetic expressions. And in Section 4, we construct a concrete computing algorithm for the verification in computer, which is an efficient computing algorithm from the view point of interval arithmetic. Finally, some numerical examples are presented in Section 5 and Section 6 for one and two dimension cases, respectively.

2 Problem and fixed point formulation

We consider the following nonlinear elliptic problem:

$$\left\{ \begin{array}{rcll} -\Delta v & = & \lambda v^+ & \text{in} \quad \Omega, \\ v & = & -1 & \text{on} \quad \partial\Omega, \end{array} \right. \tag{1}$$

where Δ is the Laplace operator, Ω a bounded and convex domain in $\mathbb{R}^n (1 \le n \le 3)$ with piecewise smooth boundary $\partial\Omega$, $\lambda \ge 0$ a real parameter, and $t^+ \equiv \max\{0, t\}$.

(1) arises in equilibrium analysis of confined MHD (magnetohydrodynamics) plasmas in an infinite cylindrical domain, which is a simplification of the so-called Grad-Shafranov equation. Roughly speaking, v and Ω correspond to the magnetic flux function and the region enclosed by a conducting shell, respectively. Another example is the equilibrium of a thin stretched membranes partially covered with water ([3]). Clearly, the function $v(x) \equiv -1$, $x \in \Omega$ satisfies (1) for all λ, and we will seek other non trivial solutions.

Remark 1. If $v(x)$ is a nontrivial solution of (1) for some λ, in a domain $\Omega_p \equiv \{x \in \Omega;\ v(x) \geq 0\}$, $w \equiv v|_{\Omega_p}$ is an eigen function of $-\Delta w = \lambda w$ in Ω_p, $w = 0$ on $\partial\Omega_p$. Then (1) may be considerd a nonlinear eigenvalue problem, and the portion of Ω where $v(x) = 0$ can be regarded as a free boundary corresponding to the boundary of plasmas.

In order to apply the formulation presented by [1], we set $v = u - 1$, $f(t) \equiv (t-1)^+$, and consider the following equivalent problem.

$$\begin{cases} -\Delta u &= \lambda f(u) \quad \text{in} \quad \Omega, \\ \quad u &= 0 \quad\quad\ \ \text{on} \quad \partial\Omega. \end{cases} \tag{2}$$

Here, let $H^m(\Omega)$ denote m-th order L^2-Sobolev space on Ω, and set

$$H_0^1(\Omega) = \{\, v \in H^1(\Omega) \mid v = 0 \ \ on \ \ \partial\Omega \,\}.$$

The inner product on $H_0^1(\Omega)$ is defined as $(\nabla u\,,\, \nabla v)$, where $(\,\cdot\,,\,\cdot\,)$ is the $L^2(\Omega)$-inner product, and set $\nabla u = (\dfrac{\partial u}{\partial x_1}, ..., \dfrac{\partial u}{\partial x_n})$. The nonlinear real-valued function $f(\cdot)$ satisties the following properties.

Lemma 1. f is the continuous map from $H_0^1(\Omega)$ to $L^2(\Omega)$.

Proof. $\forall u, v \in H_0^1(\Omega)$, and $\Omega_1 = \{\ x \in \Omega\ ;\ u(x) \geq 1, v(x) \geq 1\ \}$, $\Omega_2 = \{\ x \in \Omega\ ;\ u(x) \geq 1, v(x) < 1\}$, $\Omega_3 = \{\ x \in \Omega\ ;\ u(x) < 1, v(x) \geq 1\ \}$, following inequalities imply the continuity of f :

$$\begin{aligned} ||f(u) - f(v)||^2_{L^2(\Omega)} &= ||(u-1) - (v-1)||^2_{L^2(\Omega_1)} + ||u-1||^2_{L^2(\Omega_2)} + ||v-1||^2_{L^2(\Omega_3)} \\ &\leq ||u-v||^2_{L^2(\Omega_1)} + ||(u-1)-(v-1)||^2_{L^2(\Omega_2)} + ||(v-1)-(u-1)||^2_{L^2(\Omega_3)} \\ &\leq ||u-v||^2_{L^2(\Omega)}. \ \blacksquare \end{aligned}$$

Lemma 2. For each bounded subset U in $H_0^1(\Omega)$, $f(U)$ is also bounded in $L^2(\Omega)$.

Proof is quite similar to Lemma 1.

To verify the existence of a solution of (2) in a computer, we use the fixed point formulation. First, we rewrite (2) in the weak form:

$$\text{find} \quad u \in H_0^1(\Omega) \cap H^2(\Omega) \quad \text{s.t.} \quad (\,\nabla u\,,\, \nabla v\,) = \lambda(\,f(u)\,,\, v\,), \quad \forall v \in H_0^1(\Omega). \tag{3}$$

It is well known that for any $\psi \in L^2(\Omega)$, the boundary value problem:

$$-\Delta\phi = \psi \quad \text{in } \Omega, \qquad \phi = 0 \quad \text{on } \partial\Omega \tag{4}$$

has a unique solution $\phi \in H_0^1(\Omega) \cap H^2(\Omega)$ and the following estimate holds

$$|\,\phi\,|_{H^2(\Omega)} \leq C_1 \|\psi\|_{L^2(\Omega)}. \tag{5}$$

where C_1 is a positive constant independent of ψ, and $|\cdot|_{H^2(\Omega)}$ implies the $H^2(\Omega)$-seminorm defined by $|u|^2_{H^2(\Omega)} = \sum_{j=1}^n ||(\partial^2 u)/(\partial x_j{}^2)||^2_{L^2(\Omega)}$. For each $\psi \in L^2(\Omega)$, let $A\psi$ be the solution of (4). Then A is the compact operator from $H_0^1(\Omega)$ to $L^2(\Omega)$. Therefore, from Lemma 1 and Lemma 2, the nonlinear operator $F \equiv A\lambda f$ is a compact operator on $H_0^1(\Omega)$, and the weak solution of (3) can be rewritten as the fixed point form:

$$u = Fu.$$

Now, we will introduce a Newton-like method for the above equation. However, due to the non-smoothness of the nonlinear term $f(u)$, we can not directly apply the formulation proposed by [1]. We need an extension of the derivative of f, and a suitable modification of the Fréchet derivative of F. Let us introduce an extended derivative f' to $\mathbb{R}$ by

$$f'(t) \equiv \begin{cases} 1 & t > 1, \\ \xi & t = 1, \\ 0 & t < 1. \end{cases}$$

where ξ is a given fixed finite value. Since $f'(u)w \in L^2(\Omega)$ for all $u, w \in H^1(\Omega)$, we can define the linear operator $S(u)$ on $H_0^1(\Omega)$ for each $u \in H_0^1(\Omega)$ satisfying

$$(\nabla S(u)w, \nabla v) = \lambda(f'(u)w, v), \qquad \forall w, v \in H_0^1(\Omega).$$

We call $S(u)$ as the *Fréchet-like derivative* of F at u, and write $F'(u) \equiv S(u)$ from now on.

Next, let S_h be an appropriate finite element subspace of $H_0^1(\Omega)$ dependent on a parameter h ($0 < h < 1$) and $u_h \in S_h$ an approximate solution to (3). Further, let P_h be an orthogonal projection from $H_0^1(\Omega)$ into S_h in $H_0^1(\Omega)$-sense determined by

$$(\, \nabla(u - P_h u)\, ,\, \nabla v\,) = 0, \qquad \forall v \in S_h. \tag{6}$$

We assume that, as in [1], [2] etc.,

ASSUMPTION 1. The restriction of the operator $P_h(I - F'(u_h)) \; : \; H_0^1(\Omega) \to S_h$ to S_h has the inverse operator $[I - F'(u_h)]_h^{-1} \; : \; S_h \to S_h$, where I denotes the identity map on $H_0^1(\Omega)$.

In practice, above assumption is equivalent to the regularity of an matrix G which will be introduced in Section 4, therefore, we can check the assumption in the computational process of verification procedure.

Next, for a small parameter ε $(0 < \varepsilon < 1)$, we define the nonlinear operator T_ε on $H_0^1(\Omega)$ by

$$T_\varepsilon u \equiv \{\; I - (\; [\; I - F'(u_h)\;]_h^{-1} P_h + \varepsilon I\;)(\; I - F\;)\; \}u. \tag{7}$$

Note that if the operator $[\; I - F'(u_h)\;]_h^{-1} P_h + \varepsilon I$ is invertible, then the two fixed point forms, $u = Fu$ and $u = T_\varepsilon u$ are equivalent. We can easily show that the operator T_ε is a condensing operator, and under the Assumption 1, if there exists a non-empty, bounded, convex and closed set U in $H_0^1(\Omega)$ satisfying $T_\varepsilon U \subset U$, then T_ε has a fixed point u in U by Sadovskii's fixed point theorem.

3 Computable verification condition

The following descriptions are almost the same as in [1]. But, in order to keep the paper self-contained, we present those results. We define two concepts, the *rounding* and the *rounding error* which enable us to treat the infinite dimensional problem in a computer. Let P_h be the

H_0^1-projection defined by (6), and let $\tilde{T}_\varepsilon \equiv P_h T_\varepsilon$. For a set $U \in H_0^1(\Omega)$, we define the rounding $R(T_\varepsilon U)$ as

$$R(T_\varepsilon U) \equiv \{v \in S_h \mid v = \tilde{T}_\varepsilon u \ , \ u \in U\}.$$

Next, we assume, as the approximation property of P_h, that

ASSUMPTION 2. $\|u - P_h u\|_{H_0^1(\Omega)} \le C_2 h |u|_{H^2(\Omega)}, \quad \forall u \in H_0^1(\Omega) \cap H^2(\Omega)$, where C_2 is a positive constant independent of u and h which can be numerically determined.

This assumption holds for many finite element subspace of piecewise linear polynomials with quasi-uniform partition ([4]). Then, for $\alpha \equiv \sup_{u \in U} \|T_\varepsilon u - \tilde{T}_\varepsilon u\|_{H_0^1(\Omega)}$ and $C \equiv C_1 C_2$, we define the rounding error $RE(T_\varepsilon U)$ as

$$RE(T_\varepsilon U) \equiv \{\phi \in S_h^\perp \mid \|\phi\|_{H_0^1(\Omega)} \le \alpha \ \text{ and } \ \|\phi\|_{L^2(\Omega)} \le Ch\alpha\},$$

where $S_h^\perp$ means the orthogonal complement of S_h in $H_0^1(\Omega)$.

Now, we have the following computable verification condition. The proof is similar to that in [2].

LEMMA 3. Let $U \subset H_0^1(\Omega)$ be a non-empty, bounded, convex, and closed subset such that for some ε $(0 < \varepsilon < 1)$,

$$R(T_\varepsilon U) \ \oplus \ RE(T_\varepsilon U) \ \overset{\circ}{\subset} \ U, \tag{8}$$

then there exists a solution of $u = Fu$ in U. Here, $\oplus$ denotes the direct sum in the sense of $H_0^1(\Omega)$ and $M_1 \overset{\circ}{\subset} M_2$ implies $\overline{M_1} \subset \overset{\circ}{M_2}$ for any sets M_1, M_2.

Next, we propose a computer algorithm to construct the set U which satisfies the verification condition (8). Let $u_h \in S_h$ be a given approximation of the solution to (3), $\{\phi_j\}_{j=1}^M$ a basis of the finite element subspace S_h, where $M = \dim S_h$, and let $\mathrm{I\!I\!R}$ be the set of all closed intervals of $\mathbb{R}$. Moreover, let $S_{h,I}$ be the set of all linear combinations of $\{\phi_j\}_{j=1}^M$ with interval coefficients. That is,

$$S_{h,I} = \{w_h \subset S_h \mid w_h = \sum_{j=1}^M A_j \phi_j, \ A_j \in \mathrm{I\!I\!R}, \ (j = 1, \cdots, M) \ \}.$$

For any $\alpha \in \mathbb{R}^+$, set

$$[\alpha] \equiv \{ \ \phi \in S_h^\perp \mid \|\phi\|_{H_0^1(\Omega)} \le \alpha \ \text{ and } \ \|\phi\|_{L^2(\Omega)} \le Ch\alpha \ \}.$$

We now construct the iteration sequence $\{ \ (\delta u_h^n, \alpha_n \ )\}_{n \ge 0} \in S_{h,I} \times \mathbb{R}^+$ as below.

$n = 0$: Set $\delta u_h^0 = \{0\}$, and $\alpha_0 = 0$.

$n \ge 1$: First, for a given $0 < \sigma \ll 1$, define the *σ-inflation* of $(\delta u_h^{n-1}, \alpha_{n-1})$ by

$$\delta \bar{u}_h^{n-1} \equiv \delta u_h^{n-1} + \sum_{j=1}^M [\,-1, 1\,]\, \sigma\, \phi_j, \qquad \bar{\alpha}_{n-1} \equiv \alpha_{n-1} + \sigma$$

Next, for the set $\bar{U}^{n-1} \equiv u_h + \delta \bar{u}_h^{n-1} + [\bar{\alpha}_{n-1}]$, define $(\delta u_h^n, \alpha_n) \in S_{h,I} \times \mathbb{R}^+$ by

$$\delta u_h^n \equiv \tilde{T}_0 \bar{U}^{n-1} - u_h, \qquad \alpha_n \equiv Ch\lambda \sup_{u \in \bar{U}^{n-1}} \|f(u)\|_{L^2(\Omega)}. \tag{9}$$

Note that these iterations are carried out independent of the parameter ε. In the actual caluculation, each quantity is computed in the over estimated sense by using the interval arithmetic. Finally, the verification condition in a computer is given by following theorem (the proof is in [5]).

THEOREM 1. If for some $n \geq 1$, two relationships

$$\delta u_h^n \overset{\circ}{\subset} \delta \bar{u}_h^{n-1}, \qquad \alpha_n < \bar{\alpha}_{n-1}, \tag{10}$$

hold, then there exists a solution of (3) in $u_h + \delta u_h^n + [\alpha_n]$. Here, the first term of (10) means the strict inclusion in the sense of each coefficient interval in $\delta \bar{u}_h^{n-1}$ and δu_h^n.

4 A computational algorithm by computer

In this section, we propose the concrete computing algorithm in computer, which is the efficient computing algorithm from the view point of the interval arithmetic ([1]).

First we enclose δu_h^n in (9) as

$$\delta u_h^n = \sum_{j=1}^{M} X_j^n \phi_j, \tag{11}$$

where, $(X_j^n)_{j=1,\cdots,M}$ is an interval vector which will be defined as below. We define $\{b_j\}_{j=1,\cdots,M}$ and $\{c_j\}_{j=1,\cdots,M}$ as the coefficients of u_h and $\tilde{F}u_h \equiv P_h F u_h$ as follows:

$$u_h = \sum_{j=1}^{M} b_j \phi_j, \qquad \tilde{F}u_h = \sum_{j=1}^{M} c_j \phi_j.$$

And we denote for each $1 \leq i, j \leq M$,

$$g_{ji} \equiv (\, \nabla [I - F'(u_h)]_h \phi_j, \nabla \phi_i \,), \qquad d_{ji} \equiv (\, \nabla \phi_j, \nabla \phi_i \,),$$

and corresponding $M \times M$ matrices to the components g_{ji} and d_{ji} by G and D, respectively. From the Assumption 1, G is invertible. We also set

$$K_i^{n-1} \equiv (f(\bar{U}^{n-1}) - f(u_h) - f'(u_h)\delta \bar{u}_h^{n-1}, \phi_i).$$

Here, $(K_j^{n-1})_{j=1,\cdots,M}$ is an interval vector. Then, using the result in [1], we can prove the following proposition,

PROPOSITION 1. Interval coefficients X_i^n in (11) are determined by

$$(X_i^n) = G^{-1}(D(c_j - b_j) + \lambda (K_j^{n-1})\,). \tag{12}$$

The most important computation in the above proposition is the estimation of the interval vector $(K_j^{n-1})_{j=1,\cdots,M}$. For simplicity, we use the δu and $[\alpha]$ insted of $\delta \bar{u}_h^{n-1}$ and $[\bar{\alpha}_{n-1}]$, respectivery. Since $f(t) = (t-1)^+$, for each i $(i = 1, \cdots, M)$, K_i^{n-1} is enclosed as :

$$\begin{aligned} K_i^{n-1} &= (\, f(u_h + \delta u + [\alpha]) - f(u_h) - f'(u_h)\delta u,\ \phi_i \,) \\ &\subset (\, (u_h + \delta u - 1)^+ - (u_h - 1)^+ - f'(u_h)\delta u,\ \phi_i \,) + [-1,1] \sup_{\beta \in [\alpha]} ||\beta||_{L^2(\Omega)} ||\phi_i||_{L^2(\Omega)} \\ &\subset \kappa_i + [-1,1]\, C h \bar{\alpha}_{n-1}\, ||\phi_i||_{L^2(\Omega)}, \end{aligned}$$

where $\kappa_i = (\, (u_h + \delta u - 1)^+ - (u_h - 1)^+ - f'(u_h)\delta u,\ \phi_i \,)$. We can estimate the interval κ_i by the following result:

PROPOSITION 2. For $d \equiv ||\delta u||_{L^\infty(\Omega)}$, $\Omega_d \equiv \{x \in \Omega \mid -d+1 \le u_h \le d+1\}$,

$$\kappa_i = \int_{\Omega_d} (f(u_h + \delta u) - f(u_h) - f'(u_h)\delta u)\phi_i \, dx. \tag{13}$$

Proof. If $x \in \Omega$ satisfies $u_h(x) \ge 1 + d$, then we have

$$u_h(x) + s \ge u_h(x) - d - 1 \ge 0, \quad \forall s \in \delta u,$$

$$u_h(x) - 1 \ge d \ge 0,$$
$$f'(u_h(x)) = 1,$$

which imply $\kappa_i = 0$. Similaly, we can show that κ_i vanishes on the region of Ω such that $u_h(x) \le 1 - d$. ■

The estimate (13) for each element of Ω plays an essential role in order to enclose the width of K_i^{n-1} as narrow as possible.

5 One dimensional case

In this section, we give some numerical results for one dimensional case. We condider the following two point boundary value problem.

$$\begin{cases} -u'' = \lambda(u-1)^+ & \text{in } (0,1) \\ u(0) = u(1) = 0. \end{cases} \tag{14}$$

It is known that the problem (14) has a non-trivial solution if $\lambda > \pi^2$, and u is represented by

$$u(x) = \begin{cases} 1 + \dfrac{2\cos(\lambda^{\frac{1}{2}}(x-0.5))}{\lambda^{\frac{1}{2}} - \pi} & |x - 0.5| \le \dfrac{\pi}{2\lambda^{\frac{1}{2}}}, \\[2ex] 1 + \dfrac{\pi - 2\lambda^{\frac{1}{2}}|x-0.5|}{\lambda^{\frac{1}{2}} - \pi} & |x - 0.5| > \dfrac{\pi}{2\lambda^{\frac{1}{2}}}. \end{cases} \tag{15}$$

We divide the interval $\Omega = (0,1)$ into N equal parts and set for $i = 1, 2, \cdots, N$,

$$x_i = \frac{i}{N}, \quad \Omega_i = (x_{i-1}, x_i), \quad h = \frac{1}{N}$$

Also, let $P_1(\Omega_i)$ denote the set of linear polynomials on Ω_i and define the finite element subspace S_h by

$$S_h \equiv \{ v \in C(\Omega) \mid v|_{\Omega_i} \in P_1(\Omega_i), 1 \le i \le N, v(0) = v(1) = 0 \}. \tag{16}$$

Then, $\dim S_h = N - 1$, and we now choose the basis of S_h as the following hat functions :

$$\phi_j(x_k) = \begin{cases} 1 & (k = j), \\ 0 & (k \ne j), \end{cases} \quad \text{for } 1 \le j, k \le N.$$

We can take the constant C previously appeared as $1/\pi$ (cf. [2]).

The numerical examples for one dimensional case have been computed on an Sun Sparc 10 workstation using the library PROFIL corded by Knüppel in Technical University of Hamburg-Harburg ([6]). PROFIL is a portable C++ class fast interval library. It introduces new data types as vectors, matrices, interval vectors and matrics, etc. and lots of operations between them. Additionally, PROFIL supports an interval linear system solvers proposed by Rump ([7]). These methods compute an enclosure of the solution set x of the interval linear system $Ax = v$ where A and v are interval matrix and vector, respectively.

Now we show some numerical examples verified by the procedure in section 4.

Example 1. $\lambda = 10$, number of partitions $N = 100$, iteration numbers $n = 6$ in Theorem1, $\alpha_6 = 2.20098$.
Figure 1 shows the guarenteed intervals for an exact solution at each mesh point. Namely, it is verified that there exists a solution whose range at mesh points are included between two curves.

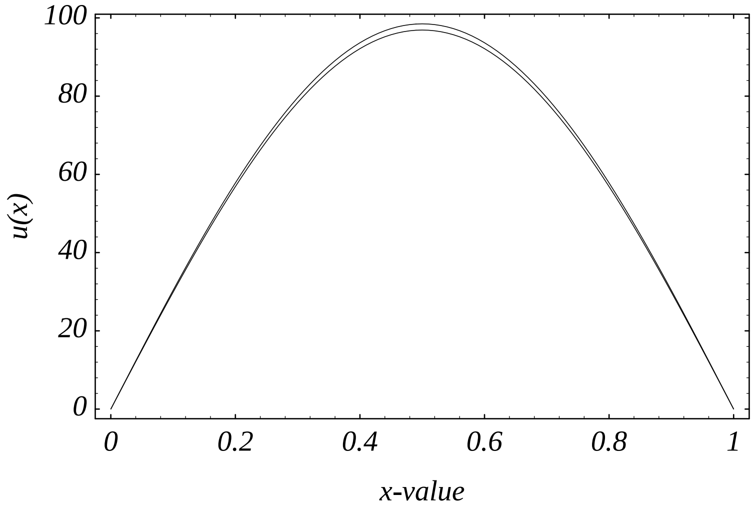

Fig. 1. Range of the solution for $\lambda = 10$

Example 2. $\lambda = 100$, number of partitions $N = 120$, iteration numbers $n = 14$, $\alpha_{14} = 0.03187$. Figure 2 shows the shape of the solution.

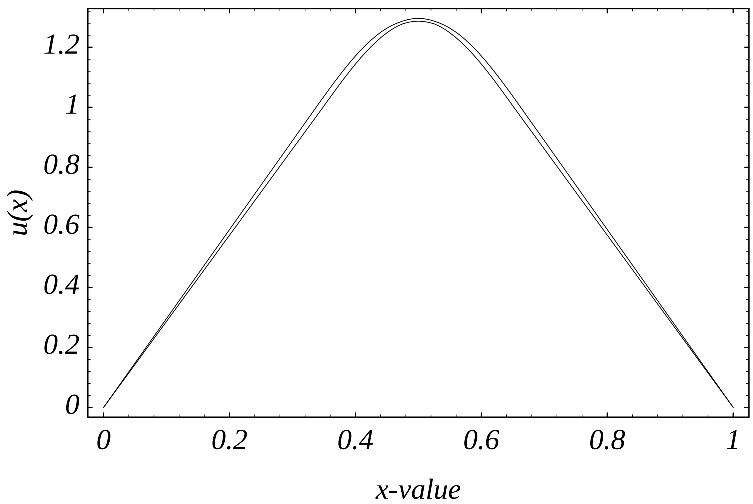

Fig. 2. Range of the solution for $\lambda = 100$

6 Two dimensional case

Let Ω be a rectangular domain in $\mathbb{R}^2$ such that $\Omega = (0,1) \times (0,1)$. Also let $\delta_x : 0 = x_0 < x_1 < \cdots < x_N = 1$ be a uniform partition, and let δ_y be the same partition as δ_x for y direction. We define the partition of Ω by $\delta \equiv \delta_x \otimes \delta_y$. Further, we define the finite element subspace S_h by $S_h \equiv \mathcal{M}_0^1(x) \otimes \mathcal{M}_0^1(y)$, where $\mathcal{M}_0^1(x)$, $\mathcal{M}_0^1(y)$ are sets of piecewise linear polynomials on $(0,1)$ defined by (16) in the variables x and y, respectively. We can also take the constant C as $1/\pi$ (cf. [8]).

In the verification process, we need to calculate the terms of the form $(f(u_h), \phi_h)$, for $u_h, \phi_h \in S_h$. These terms can be represented by

$$(f(u_h), \phi_h) = \sum_{i=1}^{L} \int_{\Omega_i} f(u_h(x))\phi_h(x),$$

where Ω_i is the rectangular element such that $\Omega = \sum_{i=1}^{L} \Omega_i$. That is, the integration can be carried out element by element. By the definition of f, for $x \in \Omega_i$,

$$f(u_h(x))\phi_h(x) = \begin{cases} (u_h(x) - 1)\phi_h(x) & u_h(x) \geq 1 \\ 0 & u_h(x) \leq 1 \end{cases}$$

Therefore, we must evaluate the portion which $u_h - 1$ changes the sign in Ω_i. The possible cases are illustrated in Figure 3.

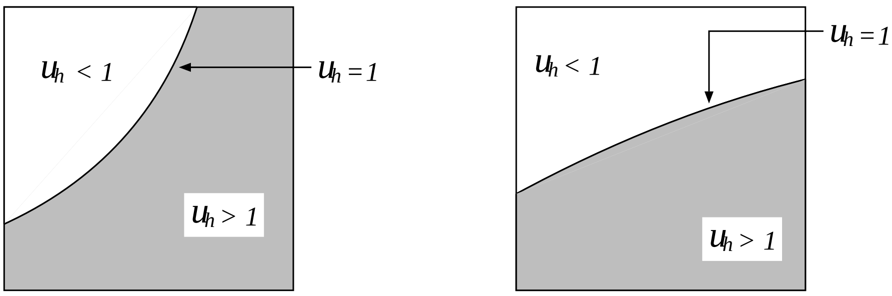

Fig. 3. Distribution of u_h in an element

For each case, using suitable change of variables, we can calculate the integration on Ω_i, and hence exact evaluation of $(f(u_h), \phi_h)$ becomes possible.

Note that we can take the approximate solution u_h such that the measure of the set in Ω satisfying $u_h = 1$ equals zero. This fact implies that we need not consider the value ξ in the definition of f'.

Example 3. $\lambda = 40$, number of partitions $N = 130$, iteration numbers $n = 5$ in Theorem1, $\alpha_5 = 0.02726$.
Figure 4 shows the outline of the shape for the solution along the line $y = 0.5$. That is, it is verified that there exists a solution between these two curves with additional $H_0^1(\Omega)$-error α_5.

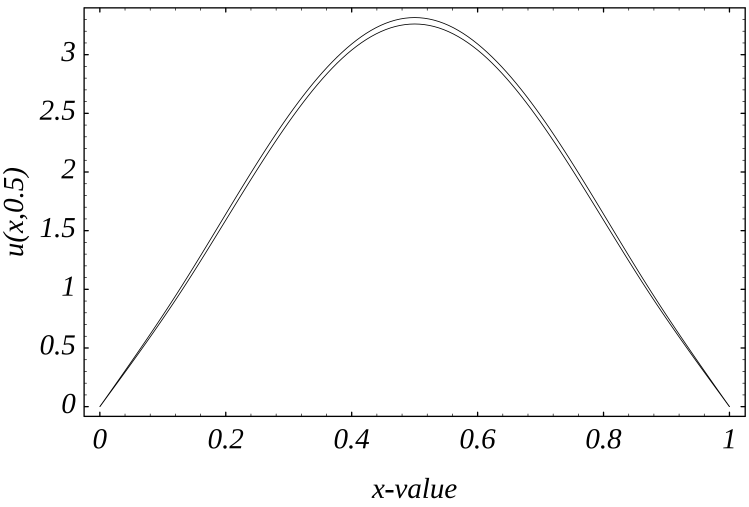

Fig. 4. Outline of the shape of the solution for $\lambda = 40$ along the line $y = 0.5$

Example 4. $\lambda = 30$, number of partitions $N = 50$, iteration numbers : 7, $\alpha_5 = 0.10956$. Figure 5 and 6 shows the approximate solution and outline of the shape for the free boundary, respectively.

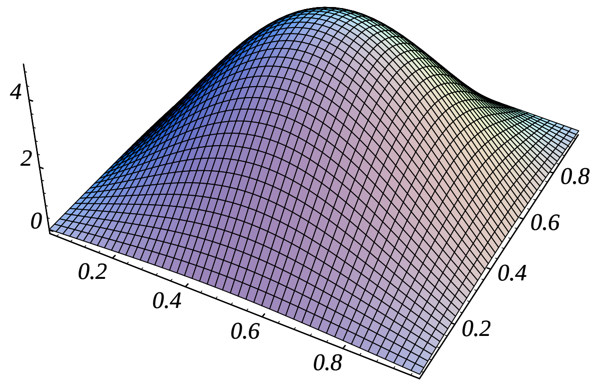

Fig. 5. approximate solution

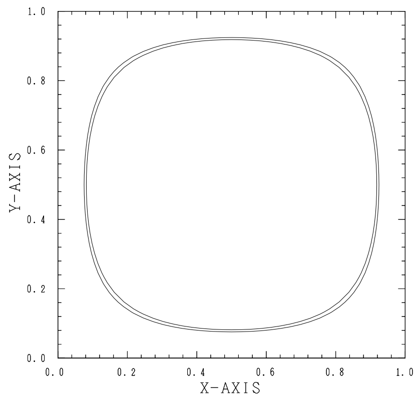

Fig. 6. Range of the free boundary

As, in any case, these calculations are too complicated to compute by hand, we used the symbolic system, *Mathematica*, which greatly reduced our efforts of computation.

The numerical examples for two dimensional case are computed on an Fujitsu VP2600/10 vector processor using computer arithmetic with double precision instead of strict interval computations. But from our experiences, the order of magnitude for the effect of round-off is under 10^{-10}. Therefore, it is almost negligible compared with the truncation error which amounts to around 10^{-2}. Figure 7 illustrates the λ - L^∞ curve obtained by our sheme. Each dot implies that we could verify the weak solutions for the corresponding λ.

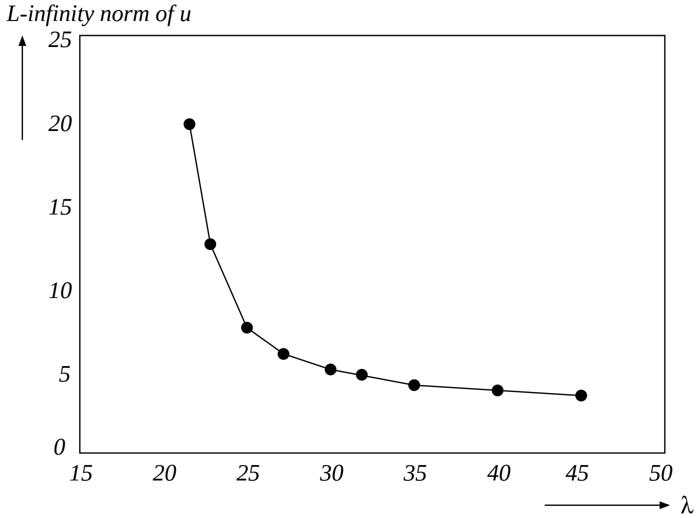

Fig. 7. λ - L^∞ curve

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