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A Numerical Verification Method of Solutions for the Navier-Stokes Equations

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Abstract. A numerical verification method of the solution for the stationary Navier-Stokes equations is described. This method is based on the infinite dimensional fixed point theorem using the Newton-like operator. We present a verification algorithm which generates automatically on a computer a set including the exact solution. Some numerical examples are also discussed.

Keywords: Navier-Stokes equations, numerical verification, fixed point theorems

1. Introduction

We proposed, in [9] and [10], a method to estimate the guaranteed a posteriori H_0^1 error bounds of the finite element solutions for the Stokes problem in mathematically rigorous sense. These papers also describe a method to derive the constructive H_0^1 a priori error estimates for the same problems based on the estimation of the largest eigenvalues for related matrices.

Furthermore, in [11], we clarified that an Aubin-Nitsche-like technique can also be applied to the constructive L^2 error estimates and establish the estimates both in a posteriori and a priori sense by using the results obtained in our previous works.

In this paper, we describe a numerical verification method of the solution for the stationary Navier-Stokes equations incorporating with a posteriori and a priori error estimates for the Stokes problem. This method is based on the method in [7], [8] for elliptic problems, but some essential extensions are necessary to deal with the convection term. Namely, special techniques are devised to overcome the difficulty from the low regularity caused by such term. We present a verification algorithm which automatically generates on a computer a set including the exact solution.

1.1. Navier-Stokes equations

We consider the following stationary Navier-Stokes equations

$$\begin{cases} -\nu\Delta u + \nabla p &= -(u\cdot\nabla)u + f \quad \text{in} \quad \Omega, \\ \operatorname{div} u &= 0 \qquad\qquad\quad\ \text{in} \quad \Omega, \\ u &= 0 \qquad\qquad\quad\ \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where Ω is a convex polygonal domain in $\mathbb{R}^2$, $u = (u_1, u_2)^T$ the two-dimensional velocity field, p a kinematic pressure field, $\nu > 0$ the viscosity constant and $f = (f_1, f_2)^T$ a pair of L^2 function on Ω which means a density of body forces per unit mass.

1.2. Some function spaces

We denote by $H^k(\Omega)$ the usual k-th order Sobolev space on Ω, and define $(\cdot,\cdot)$ as the inner product in $L^2(\Omega)$ and put

$$\begin{aligned} H_0^1(\Omega) &\equiv \{v \in H^1(\Omega)\ ;\ v = 0 \ \text{ on } \ \partial\Omega\}, \\ L_0^2(\Omega) &\equiv \{v \in L^2(\Omega)\ ;\ (v, 1) = 0\}, \\ \mathcal{S} &\equiv H_0^1(\Omega)^2 \times L_0^2(\Omega). \end{aligned}$$

The norm in $L^2(\Omega)$ and $H_0^1(\Omega)$ is denoted by $|q|_0 \equiv (q, q)^{1/2}$ and $|v|_1 \equiv |\nabla v|_0$, respectively. We also define $H^2(\Omega)$-seminorm $|\cdot|_2$ by

$$|u|_2 \equiv \left(\left|\frac{\partial^2 u}{\partial x^2}\right|_0^2 + 2\left|\frac{\partial^2 u}{\partial x \partial y}\right|_0^2 + \left|\frac{\partial^2 u}{\partial y^2}\right|_0^2 \right)^{1/2}.$$

In what follows, since no confusion may arise, we will use the same notations for the corresponding norms and inner products in $L^2(\Omega)^2$ and $H_0^1(\Omega)^2$ as in $L^2(\Omega)$ and $H_0^1(\Omega)$, respectively. Finally, we define $H^{-1}(\Omega)^2$ as the dual space of $H_0^1(\Omega)^2$ and $<\cdot,\cdot>$ as the duality pairing between $H^{-1}(\Omega)^2$ and $H_0^1(\Omega)^2$. The norm in $H^{-1}(\Omega)^2$ is denoted by $|u|_{-1}$.

2. Finite Element Approximation

We rewrite (1) in the weak form:

$$\text{find } [u,p] \in H_0^1(\Omega)^2 \times L_0^2(\Omega) \text{ such that}$$
$$\begin{cases} \nu(\nabla u, \nabla v) - (p, \operatorname{div} v) &= -((u\cdot\nabla)u, v) + (f, v) \qquad \forall v \in H_0^1(\Omega)^2, \\ (\operatorname{div} u, q) &= 0 \qquad\qquad\qquad\qquad\qquad\quad \forall q \in L_0^2(\Omega). \end{cases} \tag{2}$$

Note that, for $u \in H_0^1(\Omega)^2$, the term $(u\cdot\nabla)u$ does not belong to $L^2(\Omega)^2$ but $L^p(\Omega)^2 (p < 2)$ because of Sobolev's imbedding theorem(e.g.[2]).

Next, we introduce some finite element subspaces for the approximation of the velocity and pressure. Let $\mathcal{T}_h$ be a family of triangulations of $\Omega \subset \mathbb{R}^2$, which consist of triangles or quadrilaterals dependent on a scale parameter $h > 0$. For $\mathcal{T}_h$, we denote by $X_h \subset H_0^1(\Omega) \cap C(\bar{\Omega})$ and $Y_h \subset L_0^2(\Omega) \cap C(\bar{\Omega})$ the finite element subspaces for the approximation of each component of the velocity u and the pressure p, respectively. Here, continuity assumptions of X_h and Y_h are necessary to obtain the guaranteed error bounds for the Stokes problem(see [10]).

We set $S_h \equiv X_h^2$. Furthermore, we assume, as the approximation property of X_h, that

$$\inf_{\xi \in X_h} |v - \xi|_1 \leq C_0 h |v|_2 \qquad \forall v \in H_0^1(\Omega) \cap H^2(\Omega),$$

where C_0 is a positive constant, independent of v and h, which can be numerically determined.This assumption holds for many finite element subspaces (cf.[1]).

It is well-known, e.g.[2], that for each $\xi \in H^{-1}(\Omega)^2$, the weak form of the Stokes equation:

$$\nu(\nabla u, \nabla v) - (p, \operatorname{div} v) - (q, \operatorname{div} u) = < \xi, v > \quad \forall [v, q] \in \mathcal{S} \tag{3}$$

has a unique solution $[u, p] \in \mathcal{S}$. We suppose that for each $\xi \in H^{-1}(\Omega)^2$, there exists a unique solution $[u_h, p_h] \in S_h \times Y_h$ satisfying

$$\nu(\nabla u_h, \nabla v_h) - (p_h, \operatorname{div} v_h) - (q_h, \operatorname{div} u_h) = < \xi, v_h > \quad \forall [v_h, q_h] \in S_h \times Y_h. \tag{4}$$

The validity of this assumption can be checked by so-called discrete inf-sup condition on $S_h \times Y_h$ (cf.[2]). If $[u, p]$ is a solution of (3) and $[u_h, p_h]$ is a solution of (4), it can be easily seen that

$$|u - u_h|_1 \leq C_2 |\xi|_{-1}, \tag{5}$$

where $C_2 \equiv 2/\nu$ is a positive constant. Moreover, for the case $\xi \in L^2(\Omega)^2$, using constructive a priori error estimates described in [10], a positive constant C_1, dependent on Ω, h, C_0 and ν only, such that

$$|u - u_h|_1 \leq C_1 |\xi|_0, \tag{6}$$

can be computed.

A finite element solution $[u_h, p_h] \in S_h \times Y_h$ of problem (2) is defined by

$$\left\{ \begin{array}{rcll} \nu(\nabla u_h, \nabla v_h) - (p_h, \operatorname{div} v_h) &=& -((u_h \cdot \nabla) u_h, v_h) + (f, v_h) & \forall v_h \in S_h, \\ -(\operatorname{div} u_h, q_h) &=& 0 & \forall q_h \in Y_h. \end{array} \right. \tag{7}$$

In actual calculation, we use the interval Newton method to enclose $[u_h, p_h]$ satisfying (7) in the small intervals. In what follows, we fix the spaces X_h, S_h, Y_h and let $[u_h, p_h]$ denote an approximate solution of (2) satisfying (7). Using the component u_h of this solution we consider the following Stokes problem

$$\begin{cases} -\nu\Delta\bar{u} + \nabla\bar{p} &= -(u_h\cdot\nabla)u_h + f \quad \text{in} \quad \Omega, \\ \operatorname{div}\bar{u} &= 0 \quad \text{in} \quad \Omega, \\ \bar{u} &= 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{8}$$

By the assumption of uniqueness of the solution satisfying (4), the finite element solution $[\bar{u},\bar{p}] \in \mathcal{S}$ of (8) coincides with $[u_h, p_h]$. Consequently, setting $v_0 \in H_0^1(\Omega)^2$ and $p_0 \in L_0^2(\Omega)$ as

$$v_0 \equiv \bar{u} - u_h, \qquad p_0 \equiv \bar{p} - p_h,$$

we can compute the numerical estimates of $|v_0|_1$ and $|p_0|_0$ using a posteriori estimates for (8) because of $-(u_h\cdot\nabla)u_h + f \in L^2(\Omega)^2$ (cf.[9], [10], [11]). In what follows, v_0 is considered as an element in $H_0^1(\Omega)^2$ whose norm can be bounded, but explicit form is unknown. By (1) and (8) we have

$$\begin{cases} -\nu\Delta(u-\bar{u}) + \nabla(p-\bar{p}) &= -(u\cdot\nabla)u + (u_h\cdot\nabla)u_h, \\ \operatorname{div}(u-\bar{u}) &= 0. \end{cases}$$

Here, w and r are defined by

$$w \equiv u - \bar{u}, \qquad r \equiv p - \bar{p},$$

respectively. Then, $u = w + \bar{u} = w + v_0 + u_h$ implies the following residual form for the Navier-Stokes equation

$$\begin{cases} -\nu\Delta w + \nabla r &= g(w) \quad \text{in} \quad \Omega, \\ \operatorname{div} w &= 0 \quad \text{in} \quad \Omega, \\ w &= 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{9}$$

where

$$g(w) \equiv -((u_h + v_0 + w)\cdot\nabla)(u_h + v_0 + w) + (u_h\cdot\nabla)u_h.$$

3. Fixed Point Formulation

First, note that, since the Stokes problem

$$\begin{cases} -\nu\Delta\hat{w} + \nabla\hat{r} &= \xi \quad \text{in} \quad \Omega, \\ \operatorname{div}\hat{w} &= 0 \quad \text{in} \quad \Omega, \\ \hat{w} &= 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{10}$$

has a unique solution $[\hat{w},\hat{r}] \in \mathcal{S}$ for each $\xi \in H^{-1}(\Omega)^2$, denoting the solution $\hat{w}$ of (10) by $A\xi$, then, A is a continuous linear operator from $H^{-1}(\Omega)^2$ to $H_0^1(\Omega)^2$. Thus, setting

$$F \equiv Ag,$$

(9) is rewritten as the fixed point problem in $H_0^1(\Omega)^2$:

$$w = Fw.$$

Concerning F, the following result is obtained in [6](chapter 5):

LEMMA 1 *F is a compact map from $H_0^1(\Omega)^2$ to $H_0^1(\Omega)^2$.*

Next, from the assumption of S_h and Y_h, for any $\xi \in H^{-1}(\Omega)^2$, there exists a unique $\tilde{u}_h \in S_h$ satisfying (4). We denote this correspondence by $A_h : H^{-1}(\Omega)^2 \longrightarrow S_h$. We now define S_h^* by

$$S_h^* \equiv \{v \in H_0^1(\Omega)^2 \mid v = (A - A_h)f, \quad f \in H^{-1}(\Omega)^2\},$$

and $\bar{S}_h^*$ by the closure of S_h^* with norm $|\cdot|_1$, and we introduce the product space X by

$$X \equiv S_h \times \bar{S}_h^*$$

Then, X is a Banach space with norm

$$\max\{\,|x_h|_1, |x^*|_1\}, \quad x = [x_h, x^*] \in X.$$

We define the linear map P from X to $H_0^1(\Omega)^2$ by

$$Px \equiv x_h + x^*, \quad x = [x_h, x^*] \in X,$$

and set

$$G \equiv g \circ P.$$

Then, the map $\tilde{F} : X \longrightarrow X$ defined by

$$\tilde{F}x \equiv [A_h Gx, (A - A_h)Gx]$$

is compact because of the compactness of the map $AG = F \circ P$. Thus, if we find a nonempty, bounded, convex and closed set $W \subset X$ such that $\tilde{F}W \subset W$, then there exists a fixed point x of $\tilde{F}$ in W by Schauder's fixed point theorem. Then, for this fixed point $x = [x_h, x^*] \in X$, $Px = x_h + x^* \in H_0^1(\Omega)$ is also a fixed point of F, namely, Px is a solution of (2).

4. Newton-like Method and Computer Algorithm

Now, we introduce the Newton-like method proposed in [7], [8]. First, we define the map $\hat{g}(w) : H_0^1(\Omega)^2 \longrightarrow H^{-1}(\Omega)^2$ by

$$\hat{g}(w) \equiv -(w \cdot \nabla)w,$$

and suppose that the restriction of the operator $P_1 - A_h\hat{g}'(u_h)$: $H_0^1(\Omega)^2 \longrightarrow S_h$ to S_h has an inverse

$$[P_1 - A_h\hat{g}'(u_h)]_h^{-1} : S_h \longrightarrow S_h, \tag{11}$$

where P_1 is an H_0^1-projection from $H_0^1(\Omega)^2$ to S_h, and $\hat{g}'(u_h)$ denotes the Fréchet derivative of $\hat{g}$ at u_h. This assumption is equivalent to the invertibility of a matrix, which can be numerically checked in actual verified computations (e.g.[13]). Next, we define the Newton-like operator $N_h : X \longrightarrow S_h$ as

$$N_h x \equiv x_h - [P_1 - A_h\hat{g}'(u_h)]_h^{-1}(x_h - A_h G x), \qquad x = [x_h, x^*],$$

and the compact operator $T : X \longrightarrow X$ by

$$Tx \equiv [N_h x, (A - A_h)Gx].$$

Then, under the invertibility assumption (11), two fixed point problems: $x = Tx$ and $x = \tilde{F}x$ are equivalent.

Now, for any $v_h \in S_h$, using real coefficients $\{a_i\}_{1\le i\le 2n}$ and basis of X_h : $\{\phi_i\}_{1\le i\le n}$, we represent v_h as

$$v_h = (\sum_{i=1}^{n} a_i\phi_i, \sum_{i=1}^{n} a_{n+i}\phi_i)^T.$$

Then, we define $(v_h)_i$ by

$$(v_h)_i \equiv |a_i|, \qquad 1 \le i \le 2n.$$

Now, for any non-negative real vector $\{W_i\}_{1\le i\le 2n+2}$, we define $W_h \subset S_h$ and $W^* \subset \bar{S}_h^*$ as

$$\begin{aligned} W_h &\equiv \{w_h \in S_h \ ; \ (w_h)_i \le W_i \quad 1 \le i \le 2n\}, \\ W^* &\equiv \{w \in \bar{S}_h^* \ ; \ |w|_1 \le W_{2n+1} + W_{2n+2}\}, \end{aligned}$$

and the set $W \subset X$ as

$$W \equiv W_h \times W^*.$$

Then, we have the following computable verification condition.

THEOREM 1 *Let W_h, W^* and W be sets defined above. If the inclusions*

$$\left\{ \begin{array}{rcl} N_h(W) & \subset & W_h, \\ (A - A_h)G(W) & \subset & W^* \end{array} \right. \tag{12}$$

hold, then there exists a fixed point x of $\tilde{F}$ in W.

Proof. By the definition, W is a non-empty, closed, convex and bounded set in X. And from (12), we have $N_h(W) \times (A - A_h)G(W) \subset W_h \times W^*$ in X. Then, by the compactness of operator T, we get

$$TW \subset W \quad \text{in } X.$$

Hence, from Schauder's fixed point theorem, we obtain the desired result. □

Next, we propose a computer algorithm to construct the set W which satisfies the verification condition (12). We use the similar iterative method with inflation to that in [7], [8], etc.

First, for $N = 0$, we take appropriate initial vector $W_i^{(0)}$ $(1 \le i \le 2n+2)$ and for $\{W_i^{(0)}\}_{1\le i\le 2n+2}$, we define $W^{(0)} = W_h^{(0)} \times W^{*(0)}$. For $N \ge 1$, with a given $0 < \delta \ll 1$, we set

$$\bar{W}_i^{(N-1)} \equiv W_i^{(N-1)}(1+\delta) \qquad 1 \le i \le 2n+2,$$

and for $\{\bar{W}_i^{(N-1)}\}_{1\le i\le 2n+2}$, define the δ-*inflation* by $\bar{W}^{(N-1)} = \bar{W}_h^{(N-1)} \times \bar{W}^{*(N-1)}$.

Next, for the set $\bar{W}^{(N-1)}$, we construct the candidate set $W^{(N)} = W_h^{(N)} \times W^{*(N)}$ by

$$\left\{\begin{array}{lcl} W_h^{(N)} & \equiv & N_h \bar{W}^{(N-1)}, \\ W_{2n+1}^{(N)} & \equiv & C_1 \sup\limits_{w\in\bar{W}^{(N-1)}} |G_1(w)|_0, \\ W_{2n+2}^{(N)} & \equiv & C_2 \sup\limits_{w\in\bar{W}^{(N-1)}} |G_2(w)|_{-1}, \end{array}\right. \tag{13}$$

where for each $w = [w_h, w^*] \in X$,

$$\begin{aligned} G_2(w) &\equiv -((v_0 + w^*)\cdot\nabla)(v_0 + w^*) \ \in H^{-1}(\Omega)^2, \\ G_1(w) &\equiv G(w) - G_2(w) \ \in L^2(\Omega)^2. \end{aligned}$$

Here, $W_h^{(N)}$ is determined by the interval vector solution for the $2n$ dimensional linear system of equations with interval right-hand side (cf.[15], [16]). $W_{2n+1}^{(N)}$ corresponds to the a priori error estimates (6) for the finite element solution of the Stokes problem for the smooth part of $G(W)$, which is presented in [10]. On the other hand, $W_{2n+2}^{(N)}$ stands for the a priori estimates (5) for the solution of the Stokes problem with the less smooth (H^{-1}-element) right-hand side, which can be computed because of $G_2(w) \in L^1(\Omega)$ for all $w \in X$. For example, setting $\hat{w} = v_0 + w^*$, we have

$$(\,G_2(\hat{w}), v\,) \le |\hat{w}|_1 \|\hat{w}\|_{L^4}\|v\|_{L^4} \le C_{L^4}^2 {|\hat{w}|_1}^2 |v|_1 \quad \forall v \in H_0^1(\Omega)^2,$$

where $\|\cdot\|_{L^4}$ is the L^4-norm on Ω and C_{L^4} a explicit constant in the Sobolev imbedding thorem(see [14]). Thus we obtain $\sup_{w\in\bar{W}^{(N-1)}} |G_2(w)|_{-1} \le C_{L^4}^2 {|\hat{w}|_1}^2$. Here, in general, ${|\hat{w}|_1}^2 \ll 1$ because $\hat{w}$ is a residual part of the solution.

In the actual calculation on a computer, each quantity of (13) is computed in the over-estimated sense.

Now, we have the following verification condition in a computer.

THEOREM 2 *If, for a step K, we have*

$$W_i^{(K)} \leq \bar{W}_i^{(K-1)}, \qquad 1 \leq i \leq 2n+2,$$

then, in the set $\bar{W}^{(K-1)} = \bar{W}_h^{(K-1)} \times \bar{W}^{*(K-1)} \subset X$ *constructed by* $\{\bar{W}_i^{(K-1)}\}_{1 \leq i \leq 2n+2}$, *there exists an element x satisfying $x = Tx$.*

Proof. From Theorem 1, it is sufficient to check (12) holds for $\bar{W}^{(K-1)}$. Then, by the assumption and definition of the set $W^{(K)}$, we have

$$N_h \bar{W}^{(K-1)} \subset \bar{W}_h^{(K-1)}.$$

Next, for any $\psi \in (A - A_h)G(\bar{W}^{(K-1)})$, we can take $w \in \bar{W}^{(K-1)}$ such that

$$\begin{aligned} \psi &= (A - A_h)G(w) \\ &= (A - A_h)G_1(w) + (A - A_h)G_2(w). \end{aligned}$$

Also, by virtue of (5) and (6), we get

$$\begin{aligned} |(A - A_h)G_1(w)|_1 &\leq C_1 \sup_{w \in \bar{W}^{(K-1)}} |G_1(w)|_0 \leq \bar{W}_{2n+1}^{(K-1)}, \\ |(A - A_h)G_2(w)|_1 &\leq C_2 \sup_{w \in \bar{W}^{(K-1)}} |G_2(w)|_{-1} \leq \bar{W}_{2n+2}^{(K-1)}. \end{aligned}$$

Hence, we obtain $\psi \in \bar{W}^{*(K-1)}$, and thus

$$(A - A_h)G(\bar{W}^{(K-1)}) \subset \bar{W}^{*(K-1)}$$

holds. □

By virtue of the Newton-like operator N_h and the constructive a priori error estimate (6), the above process should be successful as the parameter h becomes small.

5. Numerical Examples

Let Ω be a rectangular domain in $\mathbb{R}^2$ such that $\Omega = (0,1) \times (0,1)$. Also let δ_x : $0 = x_0 < x_1 < \cdots < x_L = 1$ be a uniform partition in x direction, and let δ_y be the same partition as δ_x for y direction. We define the partition of Ω by $\delta \equiv \delta_x \otimes \delta_y$. L denotes the number of partitions for the interval $(0,1)$, i.e. $h = 1/L$.

Further, we define the finite element subspace X_h and Y_h by $X_h \equiv \mathcal{M}_0^2(x) \otimes \mathcal{M}_0^2(y)$ where $\mathcal{M}_0^2(x)$, $\mathcal{M}_0^2(y)$ are sets of continuous piecewise quadratic polynomials on $(0,1)$ under the above partition δ with homogeneous boundary condition, and set $Y_h \equiv \mathcal{M}_0^1(x) \otimes \mathcal{M}_0^1(y) \cap L_0^2(\Omega)$, where $\mathcal{M}_0^1(x)$, $\mathcal{M}_0^1(y)$ piecewise linear as well. By the result in [12], we can take the constant $C_0 = 1/(2\pi)$.

We choose the vector function f so that

$$\begin{aligned} u_1(x,y) &= C\sin^2\pi x\sin\pi y\cos\pi y,\\ u_2(x,y) &= -C\sin^2\pi y\sin\pi x\cos\pi x,\\ p(x,y) &= -C^2\cos 2\pi x\cos 2\pi y/16 \end{aligned}$$

are the exact solutions of (1) for an arbitrary constant C. Figure 1 and Figure 2 show the pressure and velocity field on Ω, respectively(C=1).

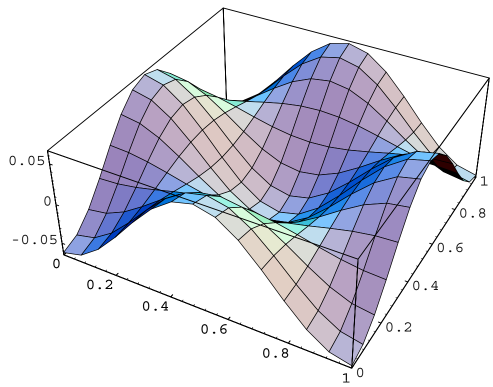

Fig. 1. Pressure field p

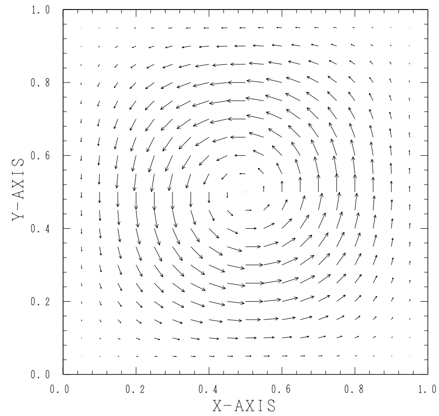

Fig. 2. Velocity field $u=(u_1,u_2)^T$.

Table 1 shows verified values of C, C_1, $|v_0|_1$, $W_{2n+1}^{(K)}$, $W_{2n+2}^{(K)}$ and $\|W_h^{(K)}\|_\infty$, where $\|\cdot\|_\infty$ stands for the L^∞-norm on Ω. Here, K is the iteration number satisfying the verification condition of Theorem 2.

$1/\nu$	L	K	C	C_1	$\|u_h\|_\infty$	$\|v_0\|_1$	$\|W_h^{(K)}\|_\infty$	$W_{2n+1}^{(K)}$	$W_{2n+2}^{(K)}$
1	10	14	1/8	0.140	7.50×10^{-2}	2.61×10^{-2}	3.52×10^{-3}	2.44×10^{-3}	1.67×10^{-4}
2	10	12	1/10	0.165	5.00×10^{-2}	1.29×10^{-2}	1.33×10^{-3}	7.67×10^{-4}	7.67×10^{-5}
5	40	22	1/10	0.143	5.00×10^{-2}	3.55×10^{-3}	1.38×10^{-3}	7.59×10^{-4}	1.89×10^{-5}
10	40	23	1/20	0.131	2.38×10^{-2}	4.19×10^{-2}	4.26×10^{-4}	1.86×10^{-4}	1.47×10^{-6}
20	40	32	1/40	0.257	1.25×10^{-2}	5.81×10^{-4}	4.45×10^{-4}	1.96×10^{-4}	2.47×10^{-6}
40	40	27	1/100	0.511	5.00×10^{-3}	4.35×10^{-4}	2.90×10^{-4}	1.08×10^{-4}	2.42×10^{-6}
50	40	13	1/200	0.639	2.50×10^{-3}	2.68×10^{-4}	8.35×10^{-5}	1.95×10^{-5}	8.48×10^{-7}
100	40	15	1/400	1.270	1.00×10^{-3}	2.09×10^{-4}	6.63×10^{-5}	1.37×10^{-5}	1.02×10^{-6}

Table 1. Verified values

Up to now, we have to choose rather small C as $1/\nu$ becomes large. The numerical examples are computed on FUJITSU VPP700/56 vector parallel processor by the usual computer arithmetic with double precision. Hence, the round off errors in these examples are neglected. However, from our experiences, the order of magnitude for the effect of round-off is under 10^{-10}. Therefore, it is almost negligible compared with the truncation error which amounts to $10^{-3} \sim 10^{-2}$. Of course, we have to use those verification software systems (e.g. [3], [4], [5]) in case that we need the rigorous mathematical proof.

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References

1. Ciarlet P. G.: *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, 1978.
2. Girault, V. and Raviart, P. A.: *Finite Element Approximation of the Navier-Stokes Equations*, Springer-Verlag, Berlin, Heidelberg, 1986.
3. Kearfott, R. B. and Kreinovich, V.: *Applications of Interval Computations*, Kluwer Academic Publishers, Netherland, 1996.
4. Knüppel, O.: PROFIL/BIAS–A Fast Interval Library, *Computing*, **53** (1994), pp. 277-287.
5. Kulisch, U., et. al.: *C-XSC, A C++ Class Library for Extended Scientific Computing*, Springer-Verlag, New York, 1993.
6. Ladyzhenskaya, O.: *The Mathematical Theory of Viscous Incompressible Flow*, Gordon & Breach, New York, 1969.
7. Nakao M. T.: A Numerical Verification Method for the Existence of Weak Solutions for Nonlinear Boundary Value Problems, *J. Math. Anal. Appl.* **164** (1992), pp. 489–507.
8. Nakao M. T.: Solving Nonlinear Elliptic Problems with Result Verification Using an H^{-1} Type Residual Iteration, *Computing, Suppl.* **9** (1993), pp. 161–173.
9. Nakao, M. T., Yamamoto, N., and Watanabe, Y.: Guaranteed Error Bounds for the Finite Element Solutions of the Stokes Problem, in: Alefeld, G., Frommer, A. and Lang B.(eds): *Scientific Computing and Validated Numerics*, Proceedings of International Symposium on Scientific Computing, Computer Arithmetic and Validated Numerics SCAN-95, Mathematical Research **90** (1996), pp. 258–264.
10. Nakao, M. T., Yamamoto, N., and Watanabe, Y.: A Posteriori and Constructive A Priori Error Bounds for Finite Element Solutions of the Stokes Equations, *J. Comput. Appl. Math.* **91** (1998), pp. 137–158.
11. Nakao, M. T., Yamamoto, N., and Watanabe, Y.: Constructive L^2 Error Estimates for Finite Element Solutions of the Stokes Equations, *Reliable Computing* **4** (1998), pp. 115–124.
12. Nakao, M. T., Yamamoto, N., and Kimura, S.: On Best Constant in the Optimal Error Estimates for the H_0^1-projection into Piecewise Polynomial Spaces, *J. Approximation Theory* **93** (1998), pp. 491–500.
13. Rump, S. M.: On the Solution of Interval Linear Systems, *Computing* **47** (1992), pp. 337–353.
14. Talenti, G.: Best Constant in Sobolev Inequality, *Ann. Math. Pura Appl.* **110** (1976), pp 353–372.
15. Watanabe, Y., and Nakao, M. T.: Numerical Verifications of Solutions for Nonlinear Elliptic Equations, *Japan J. Indust. Appl. Math.* **10** (1993) pp. 165–178.
16. Yamamoto, N.: A Numerical Verification Method for Solutions of Boundary Value Problems with Local Uniqueness by Banach's Fixed-point Thorem, *SIAM J. Numer. Anal.* **35** (1998) pp. 2004–2013.

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