Guaranteed error bounds for the finite element solutions of the Stokes problem

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0 Introduction

We describe a method to estimate the guaranteed accuracy of the finite element solutions for the Stokes problem. We show that an a posteriori error can be computed by using the numerical estimates of a constant related to the so called inf-sup condition for the continuous problem. Also a method to derive the constructive a priori error estimates are considered. Furthermore, we will mention about the numerical verification method of the solution for the stationary Navier-Stokes equation incorporating with these error estimates.

Consider the following Stokes problem:

$$\left\{ \begin{array}{rcll} -\nu\Delta u + \nabla p & = & f & \text{in} \quad \Omega, \\ \nabla\cdot u & = & 0 & \text{in} \quad \Omega, \\ u & = & 0 & \text{on} \quad \partial\Omega, \end{array} \right. \tag{0.1}$$

where $\nu > 0$ is the viscosity constant, u, f are two dimensional vectors and Ω a convex polygonal domain in $\mathbb{R}^2$. We denote by $H^k(\Omega)$ the usual k-th Sobolev space, and define $(\,\cdot,\cdot\,)$ as the inner product in $L^2(\Omega)$ and put

$$\begin{array}{rcl} H_0^1(\Omega) & \equiv & \{v \in H^1(\Omega)\ ;\ v = 0 \ \text{ on } \ \partial\Omega\}, \\ L_0^2(\Omega) & \equiv & \{v \in L^2(\Omega)\ ;\ (\,v, 1\,) = 0\}, \\ \mathcal{S} & \equiv & H_0^1(\Omega)^2 \times L_0^2(\Omega). \end{array}$$

The norm in $L^2(\Omega)$ and $H_0^1(\Omega)^2$ are denoted by $|q|_0 \equiv (\,q, q\,)^{1/2}$, $|v|_1 \equiv (|\nabla v_1|_0{}^2 + |\nabla v_2|_0{}^2)^{1/2}$, respectively. We introduce a bilinear form $\mathcal{L}$ on $\mathcal{S} \times \mathcal{S}$ by

$$\mathcal{L}(\,[u,p],[v,q]\,) \equiv \nu(\,\nabla u, \nabla v\,) - (\,p, \nabla\cdot v\,) - (\,q, \nabla\cdot u\,) \qquad [u,p],\ [v,q] \in \mathcal{S}. \tag{0.2}$$

1 Numerical estimates for inf-sup condition

Using a bilinear form $\mathcal{L}$, the standard variational formulation of (0.1) is given by :

$$\begin{array}{l} \text{find} \quad [u,p] \in \mathcal{S} \quad \text{such that} \\ \mathcal{L}(\,[u,p],[v,q]\,) = (\,f, v\,) \qquad \forall [v,q] \in \mathcal{S}. \end{array} \tag{1.1}$$

As well known, there exists a constant $\beta > 0$ depending only on Ω such that for all $q \in L_0^2(\Omega)$, there exists a $v \in H_0^1(\Omega)^2$ satisfying

$$\nabla\cdot v = q, \qquad |v|_1 \le \frac{1}{\beta}|q|_0. \tag{1.2}$$

β is a constant related to inf-sup condition for $\mathcal{L}$ which assures that problem (1.1) has a unique solution in $\mathcal{S}$. For the star shaped domains, by Horgan[1], this constant β can be numerically determind, for example, $1/\beta^2 \le 4 + 2\sqrt{2}$ for the square.

Now, using this constant β, we can describe the following lemma.

Lemma 1.1 *$\forall[u,p] \in \mathcal{S}$, let us define $\delta(u,p)$ by*

$$\delta(u,p) \equiv \sup_{[v,q]\in\mathcal{S}} \frac{\mathcal{L}([u,p],[v,q])}{|v|_1 + |q|_0},$$

then following estimates hold:

$$\begin{cases} |u|_1 \le \left(\dfrac{1}{\nu^2} + \dfrac{1}{\beta^2}\right)^{1/2} \delta(u,p), \\ |p|_0 \le \left(\dfrac{1}{\beta} + \dfrac{\nu}{\beta^2}\right) \delta(u,p). \end{cases} \tag{1.3}$$

2 Finite element approximation

Let $X_h \subset H_0^1(\Omega) \cap C(\bar{\Omega})$ and $Y_h \subset L_0^2(\Omega) \cap C(\bar{\Omega})$ be the finite element subspaces for the approximation of the velocity u and the pressure p, respectively. And set $\mathcal{S}_h \equiv X_h^2 \times Y_h$. Then the finite element solution $[u_h, p_h] \in \mathcal{S}_h$ to (1.1) is defined by

$$\nu(\nabla u_h, \nabla v_h) - (p_h, \nabla\cdot v_h) - (q_h, \nabla\cdot u_h) = (f, v_h) \qquad \forall[v_h, q_h] \in \mathcal{S}_h. \tag{2.1}$$

Now, we introduce a post-processing procedures proposed by [2]. We define X_h^* as a subspace of $H^1(\Omega)$ in which the basis of X_h^* are the sum of the basis of S_h and base functions corresponding to nodes on the boundary $\partial\Omega$. Note that

$$X_h \subset X_h^* \subset H^1(\Omega), \quad X_h \ne X_h^*.$$

We also denote P_0 as a L^2-projection from $L^2(\Omega)$ to X_h, $\hat{P}_0$ as a L^2-projection from $L^2(\Omega)$ to X_h^* and P_1 as a H_0^1-projection from $H_0^1(\Omega)$ to X_h. For $w_h \in X_h$, we denote $\overline{\nabla} w_h \in (X_h^*)^2$ and $\overline{\Delta} w_h \in L^2(\Omega)$ by

$$\begin{aligned} \overline{\nabla} w_h &\equiv (\hat{P}_0 \frac{\partial w_h}{\partial x}, \hat{P}_0 \frac{\partial w_h}{\partial y}), \\ \overline{\Delta} w_h &\equiv \nabla\cdot\overline{\nabla} w_h. \end{aligned}$$

It is easily shown that for all $v_h \in X_h^2$, the following properties hold:

$$(-\overline{\Delta} v_h, \phi) = (\overline{\nabla} v_h, \nabla \phi) \qquad \forall \phi \in H_0^1(\Omega)^2. \tag{2.2}$$

$$|\overline{\nabla} w_h - \nabla w_h|_0 = \inf_{w \in (X_h^*)^2} |w - \nabla w_h|_0 \qquad \forall w_h \in X_h. \tag{2.3}$$

Now, we assume, as the approximation property of X_h, that

$$\inf_{\xi \in X_h} |v - \xi|_1 \leq C_0 h |v|_2 \qquad \forall v \in H_0^1(\Omega) \cap H^2(\Omega), \tag{2.4}$$

where C_0 is a positive constant independent of v and h which can be numerically determined. From the properties of projection P_1 and (2.4), we obtain the following estimates.

$$|v - P_1 v|_0 \leq C_0 h |v|_1 \qquad \forall v \in H_0^1(\Omega). \tag{2.5}$$

3 A posteriori error estimates

Let (u, p) and (u_h, p_h) be the solutions of (1.1) and (2.1), respectively. We denote $e_h \in H_0^1(\Omega)$ and $\varepsilon_h \in L_0^2(\Omega)$ as the error of velocity and pressure by

$$\begin{cases} e_h \equiv u - u_h, \\ \varepsilon_h \equiv p - p_h, \end{cases}$$

respectively. From (2.2) and (2.4), we have the following lemma:

Lemma 3.1 *For all $[v, q] \in \mathcal{S}$,*

$$\frac{\mathcal{L}([e_h, \varepsilon_h], [v, q])}{|v|_1 + |q|_0} \leq \nu |\overline{\nabla} u_h - \nabla u_h|_0 + C_0 h |\nu \overline{\Delta} u_h - \nabla p_h + f|_0 + |\nabla \cdot u_h|_0. \tag{3.1}$$

Thus, we obtain the following a posteriori error estimates:

Theorem 3.1 (a posteriori error estimate)

$$\begin{cases} |u - u_h|_1 \leq \left(\dfrac{1}{\nu^2} + \dfrac{1}{\beta^2} \right)^{1/2} C(u_h, p_h), \\ |p - p_h|_0 \leq \left(\dfrac{1}{\beta} + \dfrac{\nu}{\beta^2} \right) C(u_h, p_h), \end{cases} \tag{3.2}$$

where $C(u_h, p_h)$ is an a posteriori error estimator which can be computed from the finite element solutions (u_h, p_h), C_0 and f by

$$C(u_h, p_h) \equiv \nu |\overline{\nabla} u_h - \nabla u_h|_0 + C_0 h |\nu \overline{\Delta} u_h - \nabla p_h + f|_0 + |\nabla \cdot u_h|_0. \tag{3.3}$$

By virtue of (2.1) and (2.3), it is expected that each term in the right hand side of (3.1) tends to be smaller as making h smaller.

4 A priori error estimate

From (1.1), Green's formula and (2.4), we also get the following lemma:

Lemma 4.1 $\forall [v,q] \in \mathcal{S}$,

$$\frac{\mathcal{L}([e_h, \varepsilon_h], [v,q])}{|v|_1 + |q|_0} \leq C_0 h |f - \nabla p_h|_0 + |\nabla \cdot u_h|_0. \tag{4.1}$$

We estimate the right hand side of (4.1). If we can take the positive constants K_1 and K_2 such that

$$|-\nabla p_h + P_0 f|_0 \leq K_1 |P_0 f|_0, \tag{4.2}$$

$$|\nabla \cdot u_h|_0 \leq K_2 |P_0 f|_0 \tag{4.3}$$

independent of $f \in L^2(\Omega)$, then the following a priori estimates hold

Theorem 4.1 (a priori error estimate) *For all* $f \in L^2(\Omega)^2$,

$$\begin{cases} |u - u_h|_1 \leq \left(\dfrac{1}{\nu^2} + \dfrac{1}{\beta^2}\right)^{1/2} C'(h)\,|f|_0, \\ |p - p_h|_0 \leq \left(\dfrac{1}{\beta} + \dfrac{\nu}{\beta^2}\right) C'(h)\,|f|_0, \end{cases}$$

where $C'(h)$ *is a positive constant which can be computed from* K_1, K_2, h *and* C_0 *independent of* (u_h, p_h) *by*

$$C'(h) = \sqrt{(C_0 h K_1 + K_2)^2 + (C_0 h)^2}. \tag{4.4}$$

Now we briefly describe how to estimate K_1 and K_2. Let us denote the basis of X_h as ϕ_j $(j = 1, \cdots, n,\ n = \dim X_h)$, $f = (f_1, f_2)$ and $g \in \mathbb{R}^{2n}$ as

$$g \equiv ((f_1, \phi_1), \cdots, (f_1, \phi_n), (f_2, \phi_1), \cdots, (f_2, \phi_n))^T.$$

Then, each term in (4.2) and (4.3) can be represented by quadratic forms of $2n$-dimensional vectors g as follow:

$$\begin{aligned} |-\nabla p_h + P_0 f|_0{}^2 &= g^T A g, \\ |\nabla \cdot u_h|_0{}^2 &= g^T B g, \\ |P_0 f|_0{}^2 &= g^T L g, \end{aligned}$$

where, A, B are $2n \times 2n$ symmetric matrix, L a $2n \times 2n$ is a positive definite and symmetric matrix. Hence, K_1 and K_2 can be estimated as follows.

$$K_1 \leq \left(\sup_{x \in \mathbb{R}^{2n}} \frac{x^T A x}{x^T L x}\right)^{1/2},$$

$$K_2 \leq \left(\sup_{x \in \mathbb{R}^{2n}} \frac{x^T B x}{x^T L x}\right)^{1/2}.$$

Therefore, the estimation of these values is reduced to the matrix eigenvalue problem.

5 Numerical examples

Let Ω be a rectangular domain in $\mathbb{R}^2$ such that $\Omega = (0,1) \times (0,1)$. Also let δ_x : $0 = x_0 < x_1 < \cdots < x_N = 1$ be a uniform partition, and let δ_y be the same partition as δ_x for y direction. We define the partition of Ω by $\delta \equiv \delta_x \otimes \delta_y$. Further, we define the finite element subspace X_h and Y_h by $X_h \equiv \mathcal{M}_0^2(x) \otimes \mathcal{M}_0^2(y)$ where $\mathcal{M}_0^2(x)$, $\mathcal{M}_0^2(y)$ are sets of piecewise quadratic polynomials on $(0,1)$ with homogeneous boundary condition and set $Y_h \equiv \mathcal{M}_0^1(x) \otimes \mathcal{M}_0^1(y)$ where $\mathcal{M}_0^1(x)$, $\mathcal{M}_0^1(y)$ piecewise linear as well. We can also take the constant $\nu = 1$, $C_0 = 1/(2\pi)$ ([3]) and $1/\beta^2 = 4 + 2\sqrt{2}$.

Figure 1 illustrates the following a priori error constants for the velocity and pressure in Theorem 4.1:

$$\left(\frac{1}{\nu^2} + \frac{1}{\beta^2}\right)^{1/2} C'(h) \quad \text{and} \quad \left(\frac{1}{\beta} + \frac{\nu}{\beta^2}\right) C'(h),$$

respectively.

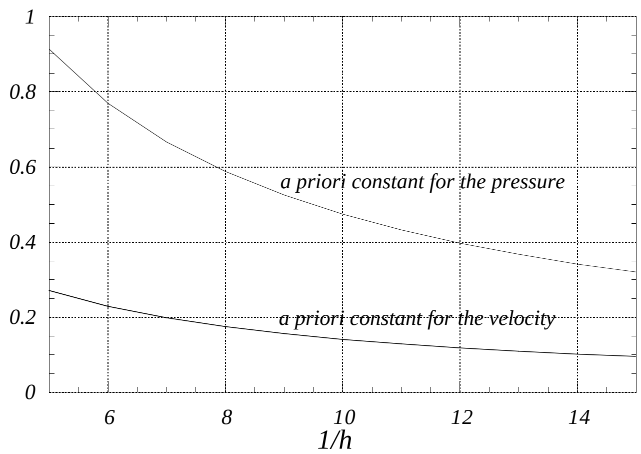

Fig. 1. A priori error constants

Next we take the vector function f so that $u = (u_1, u_2)$:

$$\begin{aligned} u_1(x,y) &= 20x^2(1-x)^2 y(1-y)(1-2y), \\ u_2(x,y) &= 20y^2(1-y)^2 x(1-x)(1-2x), \end{aligned}$$

and

$$p(x,y) = 4x(-1+2y)(10x^2 - 15x^3 + 6x^4 - 10y + 30xy - 20x^2y + 10y^2 - 30xy^2 + 20x^2y^2)$$

are the exact solutions for (0.1). In this case, $|u|_1 = 4/7 \sim 0.571$ and $|p|_0 = 2\sqrt{962/33}/7 \sim 1.543$. Figure 2 shows the vector field $u = (u_1, u_2)$ on Ω.

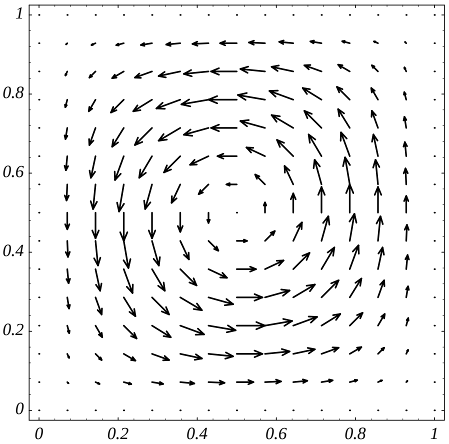

Fig.2. vector field u

Figure 3 illustrates each relative error bound from Theorem 3.1 defined by

$$\frac{|u - u_h|_1}{|u|_1} \quad \text{and} \quad \frac{|p - p_h|_0}{|p|_0},$$

respectively.

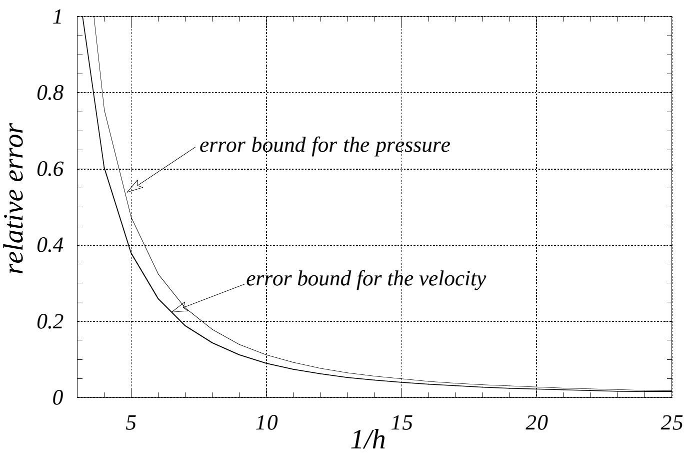

Fig. 3. a posteriori error bounds

The numerical examples are computed on FUJITSU VP2600/10 vector processor by the usual computer arithmetic with double precision. So, the round off errors in these examples are neglected. However, it should be sufficient that these results confirms us the expected order of the error.

References

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