Rayleigh-Bénard対流の定常解に対する精度保証付き数値計算 II

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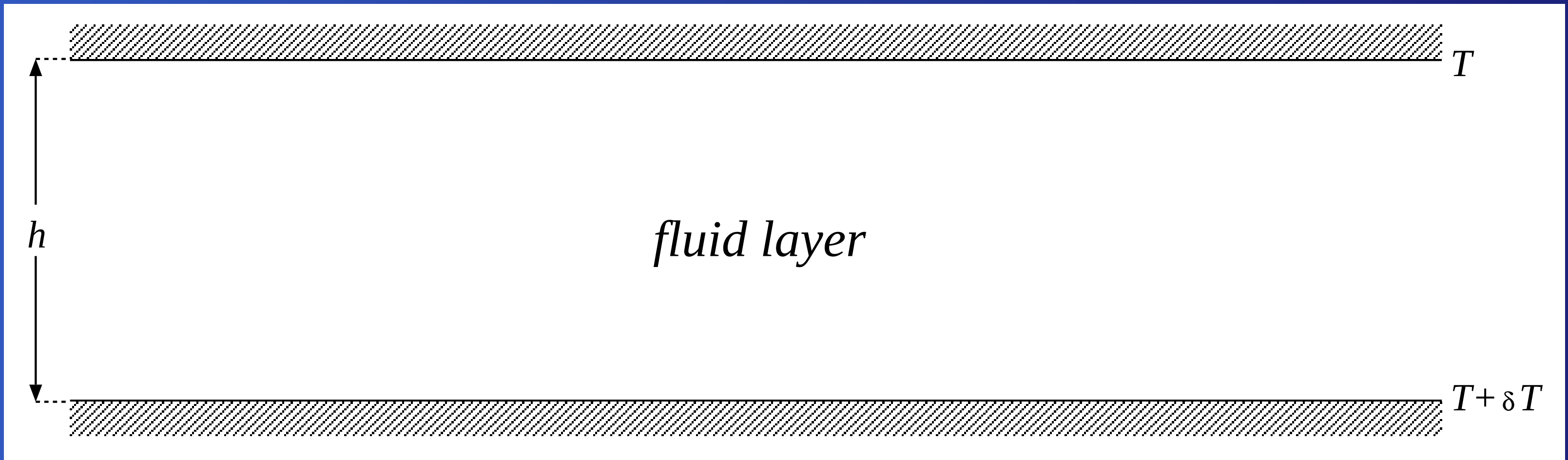
The Rayleigh-Bénard Problem

In 1916, Lord Rayleigh (Strutt, John William; 1842–1919) considered the problem of the onset of thermal convection in a plane horizontal layer ($0 \leq z \leq h$) of fluid heated from below*; T at $z = h$ and $T + \delta T$ at $z = 0$.



*Rayleigh, J.W.S.: “On convection currents in a horizontal layer of fluid, when the higher temperature is on the under side,” *The London, Edinburgh and Dublin Philosophical Magazine and Journal of Science, Ser.6*, Vol.32, pp.529–546 (1916).

Oberbeck-Boussinesq Equations

$$\begin{cases} u_t + uu_x + wu_z &= p_x + \mathcal{P}\Delta u, \\ w_t + uw_x + ww_z &= p_z - \mathcal{P}\mathcal{R}\,\theta + \mathcal{P}\Delta w, \\ u_x + w_z &= 0, \\ \theta_t + w + u\theta_x + w\theta_z &= \Delta\theta. \end{cases}$$

$(u, 0, w)$: velocity field,

p : pressure,

θ : temperature,

$\mathcal{R}$: Rayleigh number,

$\mathcal{P}$: Prandtl number.

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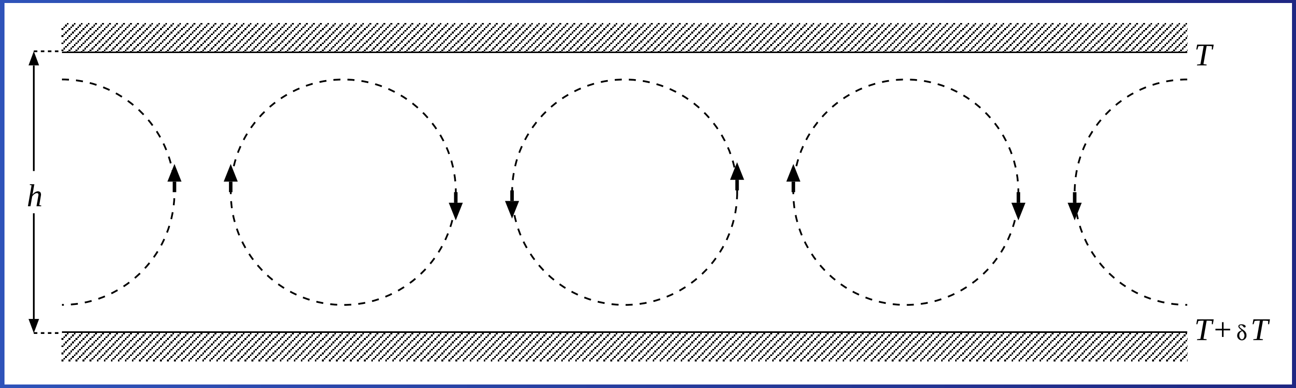

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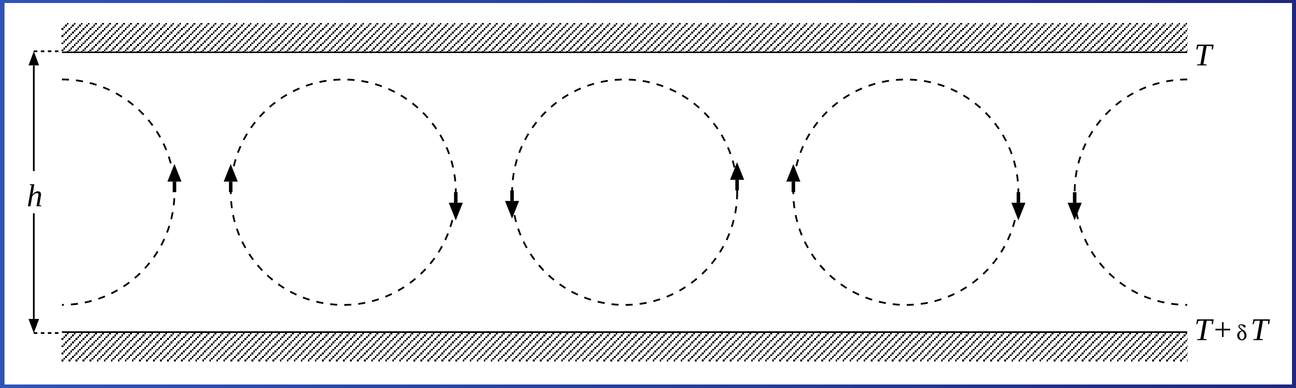

Fluid	Prandtl number
Mercury	0.025
Air	0.71
Water	6.7
Freon 113	7
Silicone oil	860

Theoretical Results

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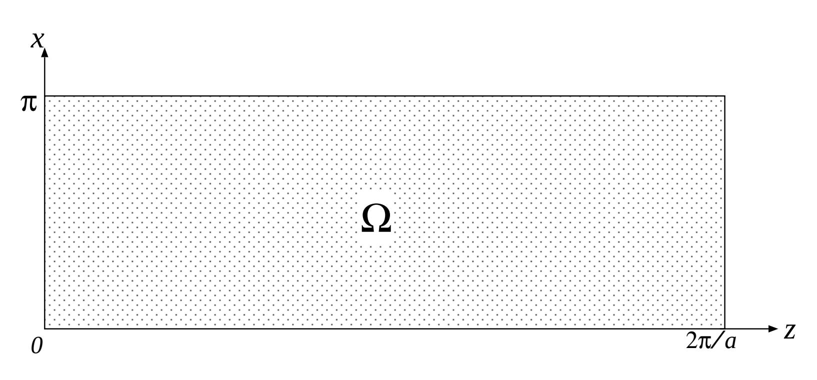
Global structure of bifurcated solutions after the critical Rayleigh point $\mathcal{R}_{\mathcal{C}}$ has not been known *theoretically* up to now.

Imposed Conditions

- Restrict the problem to the rectangular region:

$$\Omega := \{0 < x < 2\pi/a, \quad 0 < z < \pi\},$$



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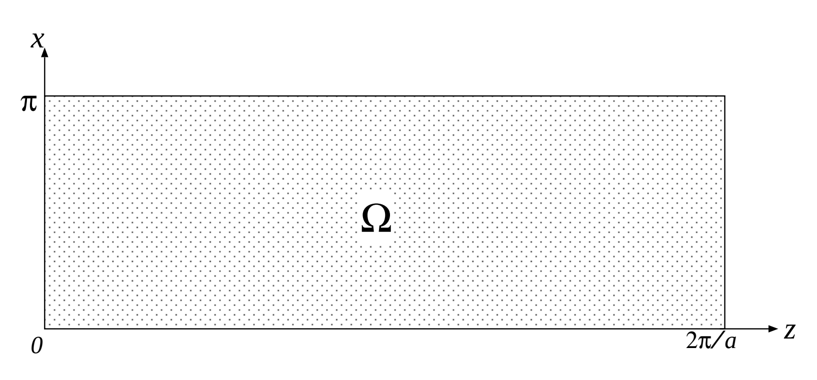

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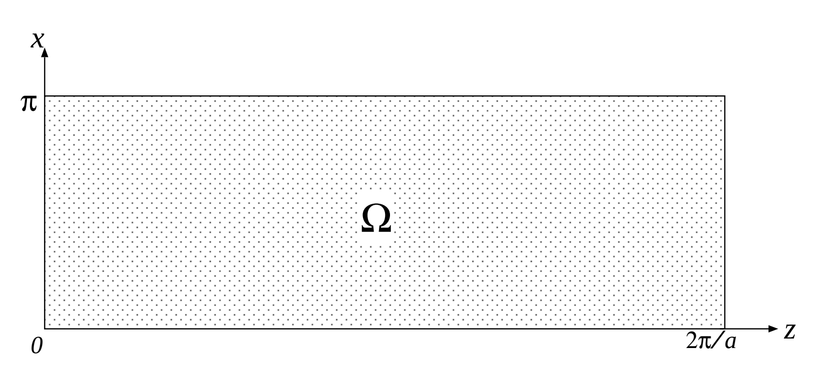

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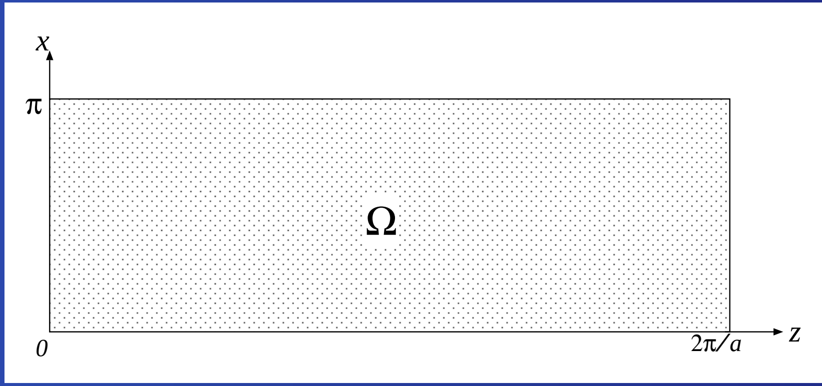

- periodic boundary conditions (period $2\pi/a$) in the horizontal direction
- stress-free boundary conditions ($u_z = w = 0$) on the surfaces $z = 0$, $z = \pi$
- evenness and oddness conditions:

$$u(x,z) = -u(-x,z), \quad w(x,z) = w(-x,z), \quad \theta(x,z) = \theta(-x,z)$$

The Stream Function

Introduce the stream function Ψ, through the definitions:

$$u = -\Psi_z, \qquad w = \Psi_x$$

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$$\begin{cases} -\Delta\Psi_t + \mathcal{P}\Delta^2\Psi &= \sqrt{\mathcal{PR}}\Theta_x - \Psi_z\Delta\Psi_x + \Psi_x\Delta\Psi_z, \\ \Theta_t - \Delta\Theta &= -\sqrt{\mathcal{PR}}\Psi_x + \Psi_z\Theta_x - \Psi_x\Theta_z. \end{cases}$$

Stationary Problems

Find the steady-state solutions to the following system:

$$\begin{cases} \mathcal{P}\Delta^2\Psi = \sqrt{\mathcal{P}\mathcal{R}}\,\Theta_x - \Psi_z\Delta\Psi_x + \Psi_x\Delta\Psi_z & \text{in } \Omega, \\ -\Delta\Theta = -\sqrt{\mathcal{P}\mathcal{R}}\,\Psi_x + \Psi_z\Theta_x - \Psi_x\Theta_z & \text{in } \Omega, \\ \quad + \text{ imposed conditions.} \end{cases}$$

$\Omega = \{0 < x < 2\pi/a,\ 0 < z < \pi\}, \quad a > 0,$

$\Psi(x, z)$: stream function,

$\Theta(x, z)$: deviation of the temperature from the linear profile,

$\mathcal{P}$: Prandtl number,

$\mathcal{R}$: Rayleigh number.

Fourier Expansions

Ψ and Θ can be represented by the following double Fourier expansions because of the imposed condition:

$$\begin{cases} \Psi(x,z) = \displaystyle\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin(amx)\sin(nz), \\ \Theta(x,z) = \displaystyle\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} B_{mn}\cos(amx)\sin(nz). \end{cases}$$

$$X^k := \left\{ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin(amx)\sin(nz) \;\middle|\; \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}((am)^{2k}+n^{2k})A_{mn}^2 < \infty \right\}.$$

$$Y^k := \left\{ \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} B_{mn}\cos(amx)\sin(nz) \;\middle|\; \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}((am)^{2k}+n^{2k})B_{mn}^2 < \infty \right\}$$

Approximate Subspaces

$$S_N^{(1)} := \left\{ \sum_{m=1}^{M_1} \sum_{n=1}^{N_1} A_{mn} \sin(amx) \sin(nz) \right\},$$

$$S_N^{(2)} := \left\{ \sum_{m=0}^{M_2} \sum_{n=1}^{N_2} B_{mn} \cos(amx) \sin(nz) \right\}.$$

Projection $P_N = (P_N^{(1)}, P_N^{(2)}) : X^4 \times Y^2 \longrightarrow S_N^{(1)} \times S_N^{(2)}$

$$\begin{cases} (\Delta^2(P_N^{(1)}\Psi - \Psi), v_N^{(1)})_{L^2} = 0 & \forall v_N^{(1)} \in S_N^{(1)}, \\ (\Delta(P_N^{(2)}\Theta - \Theta), v_N^{(2)})_{L^2} = 0 & \forall v_N^{(2)} \in S_N^{(2)}, \end{cases}$$

where $(\,\cdot,\ \cdot)_{L^2}$ means the inner product on $L^2(\Omega)$.

A Priori Estimates I

$$\Delta^2 \Psi = g^{(1)}$$

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If $N := M_1 = N_1$, $a = 1/\sqrt{2}$ and $\|\cdot\|_0 := \|\cdot\|_{L^2(\Omega)}$,

$$\begin{aligned}
\|\Psi - P_N^{(1)}\Psi\|_0 &\le \frac{1}{((N+1)^2/2+1)^2}\|g^{(1)}\|_0 \sim O(1/N^4), \\
\|\nabla(\Psi - P_N^{(1)}\Psi)\|_0 &\le \frac{N+1}{((N+1)^2/2+1)^2}\|g^{(1)}\|_0 \sim O(1/N^3), \\
\|\Delta(\Psi - P_N^{(1)}\Psi)\|_0 &\le \frac{N+1}{((N+1)^2/2+1)^{3/2}}\|g^{(1)}\|_0 \sim O(1/N^2), \\
\|\nabla(\Delta(\Psi - P_N^{(1)}\Psi))\|_0 &\le \frac{N+1}{(N+1)^2/2+1}\|g^{(1)}\|_0 \sim O(1/N).
\end{aligned}$$

A Priori Estimates II

Moreover, L^∞-estimates can be obtained, for example,

$$\begin{aligned}\|\Psi - P_N^{(1)}\Psi\|_{L^\infty(\Omega)} &\leq 0.18927\|\Psi - P_N^{(1)}\Psi\|_0 + 1.1894\|\nabla(\Psi - P_N^{(1)}\Psi)\|_0 \\ &+1.4430\|\Delta(\Psi - P_N^{(1)}\Psi)\|_0\end{aligned}$$

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$$\boxed{-\Delta\Theta = g^{(2)}} \qquad \left\{\begin{aligned}\Theta(x,z) &= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} B_{mn}\cos(amx)\sin(nz), \\ P_N^{(2)}\Theta(x,z) &= \sum_{m=0}^{M_2}\sum_{n=1}^{N_2} B_{mn}\cos(amx)\sin(nz),\end{aligned}\right.$$

If $N := M_2 = N_2$, $a = 1/\sqrt{2}$,

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- Set a candidate set U and check the criterion

$$\begin{cases} P_N F U & \subset & P_N U \\ (I - P_N)FU & \subset & (I - P_N)U \end{cases}$$

Numerical Environment

Specification of numerical environment

machines	Alpha Server GS320
OS	Tru64 UNIX V5.1
software	Fortran V5.4-1283
performance	Alpha 21264 731MHz

- In verification step, interval arithmetic is used to take account of the effects of rounding errors in the floating point computations. We use Fortran 90 library INTLIB90 coded by R.B.Kearfott.
- Kearfott, R. B., and Kreinovich, V., *Applications of Interval Computations*, Kluwer Academic Publishers, Netherland, 1996. (`http://interval.usl.edu/kearfott.html`)

The Stationary Bifurcation

Rayleigh (1916) considered the linearized stability and found the critical Rayleigh number:

$$\mathcal{R}_{\mathcal{C}} = \inf_{m,n} \frac{(a^2m^2 + n^2)^3}{a^2m^2} = 6.75 \quad (m = 1, n = 1, a = 1/\sqrt{2}).$$

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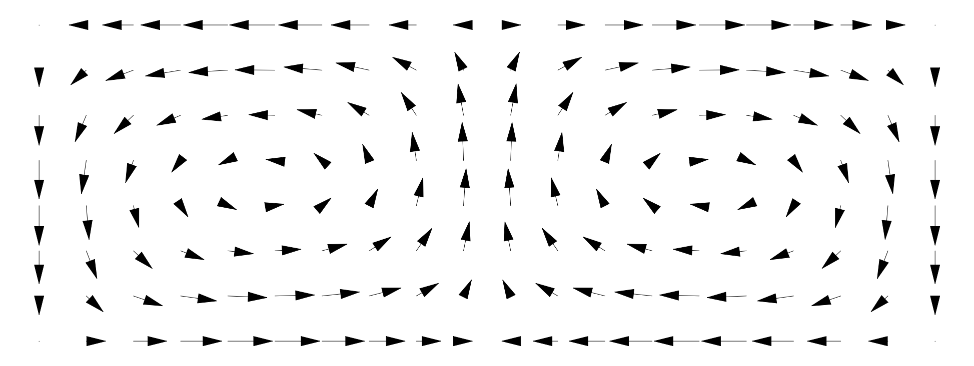
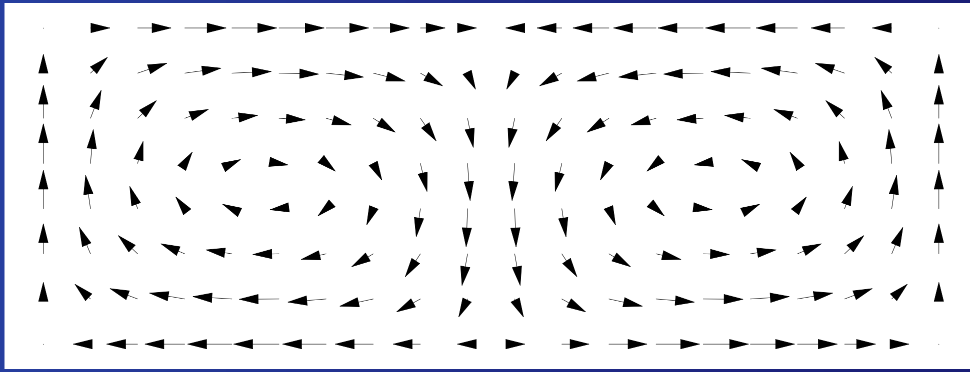
$$\hat{\Psi}_N = \sum_{m=1}^{M_1}\sum_{n=1}^{N_1} \hat{A}_{mn}\sin(amx)\sin(nz),\ \mathcal{P}=10,\ \mathcal{R}=50,\ M_1=N_1=10.$$



$\hat{A}_{11} \approx 15.37$



$\hat{A}_{11} \approx -15.37$

The First Bifurcated Solutions I

The temperature

$$\theta = \delta T(1 - z/\pi - \Theta/\sqrt{\mathcal{RP}}\pi) + T, \qquad T = 0,\ \delta T = 5,\ \mathcal{P} = 10$$

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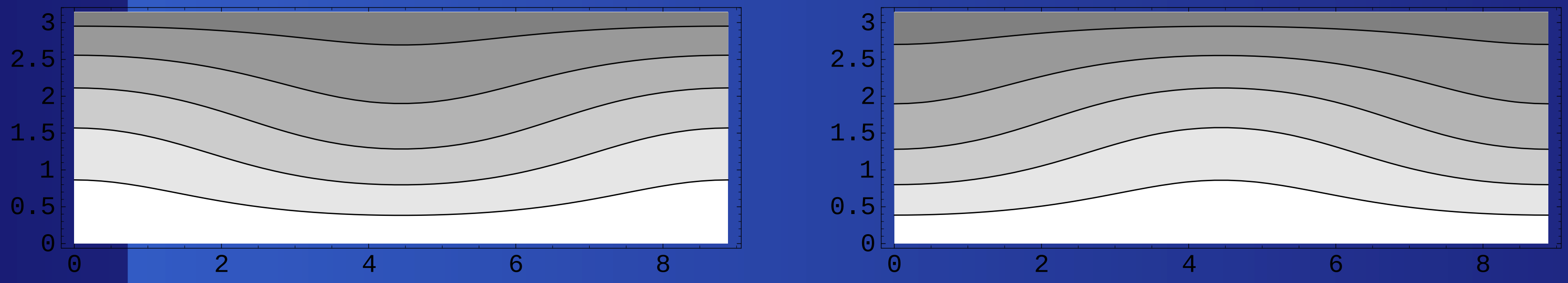

$\mathcal{R} = 7$

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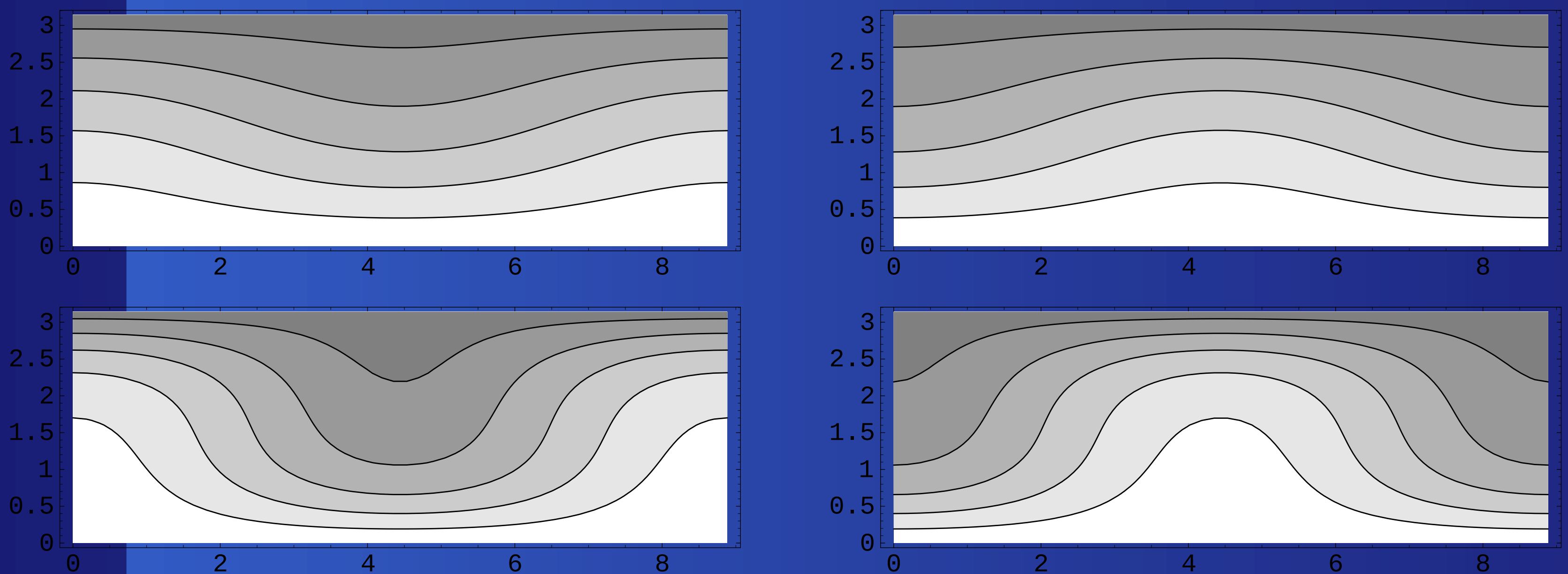

$\mathcal{R} = 7$

$\mathcal{R} = 10$

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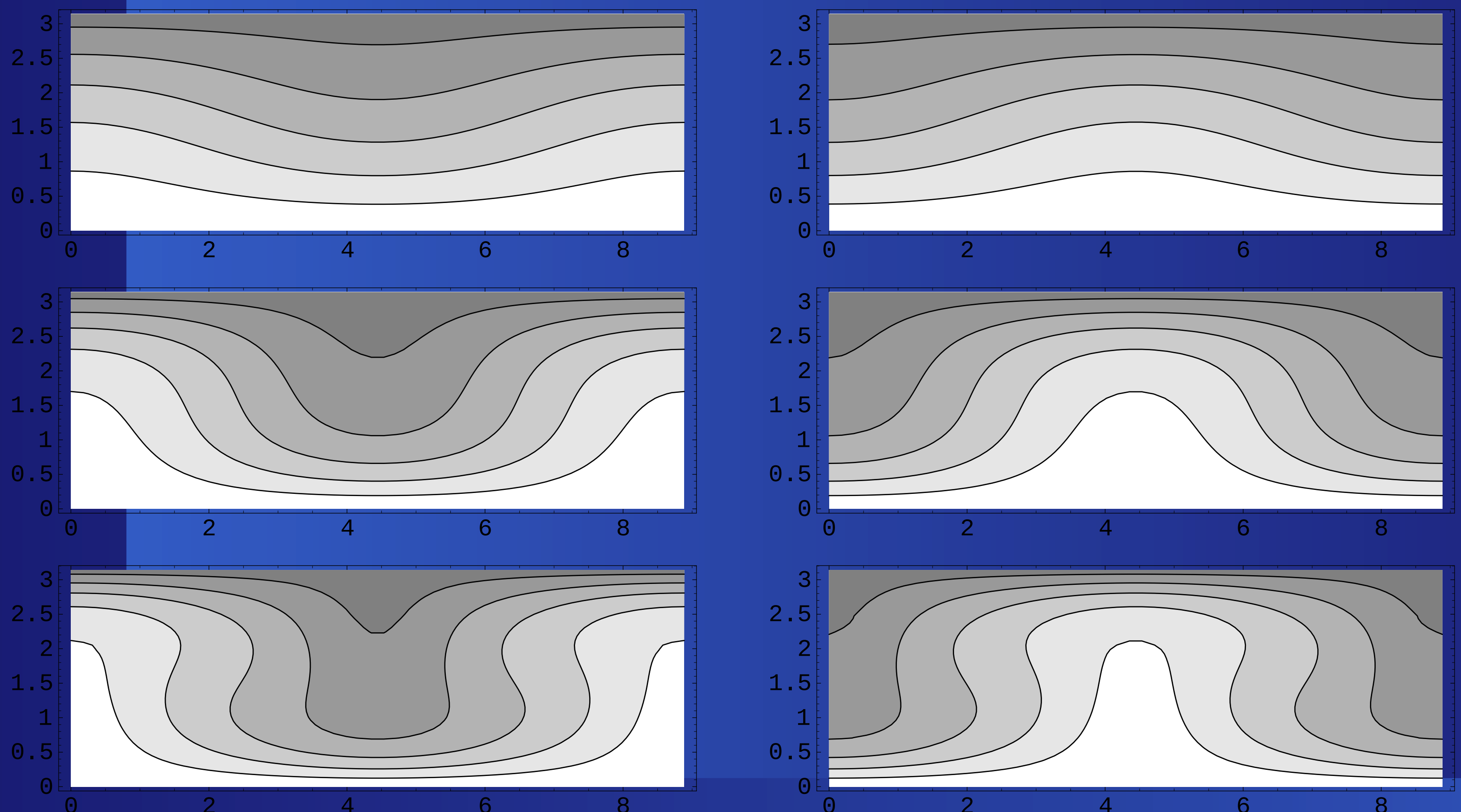

$\mathcal{R} = 7$

$\mathcal{R} = 10$

$\mathcal{R} = 20$

The First Bifurcated Solutions II

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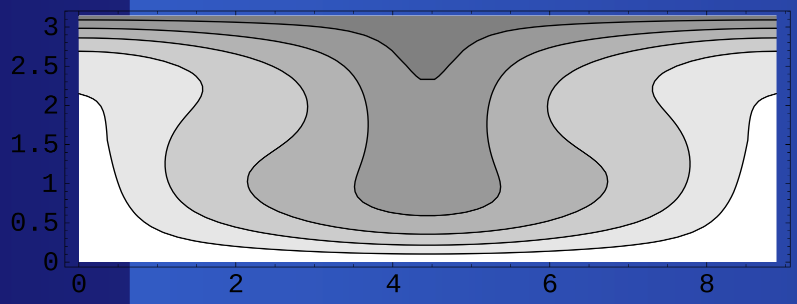

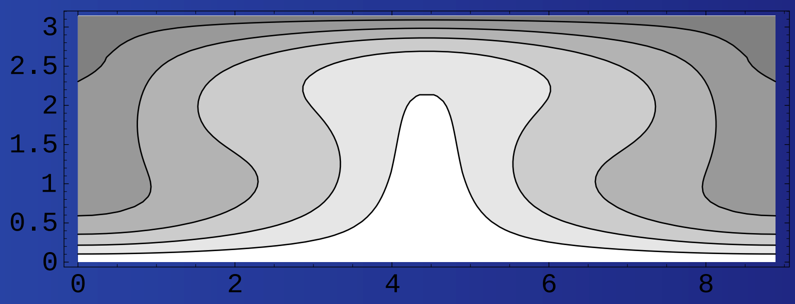

$\mathcal{R} = 30$

The First Bifurcated Solutions II

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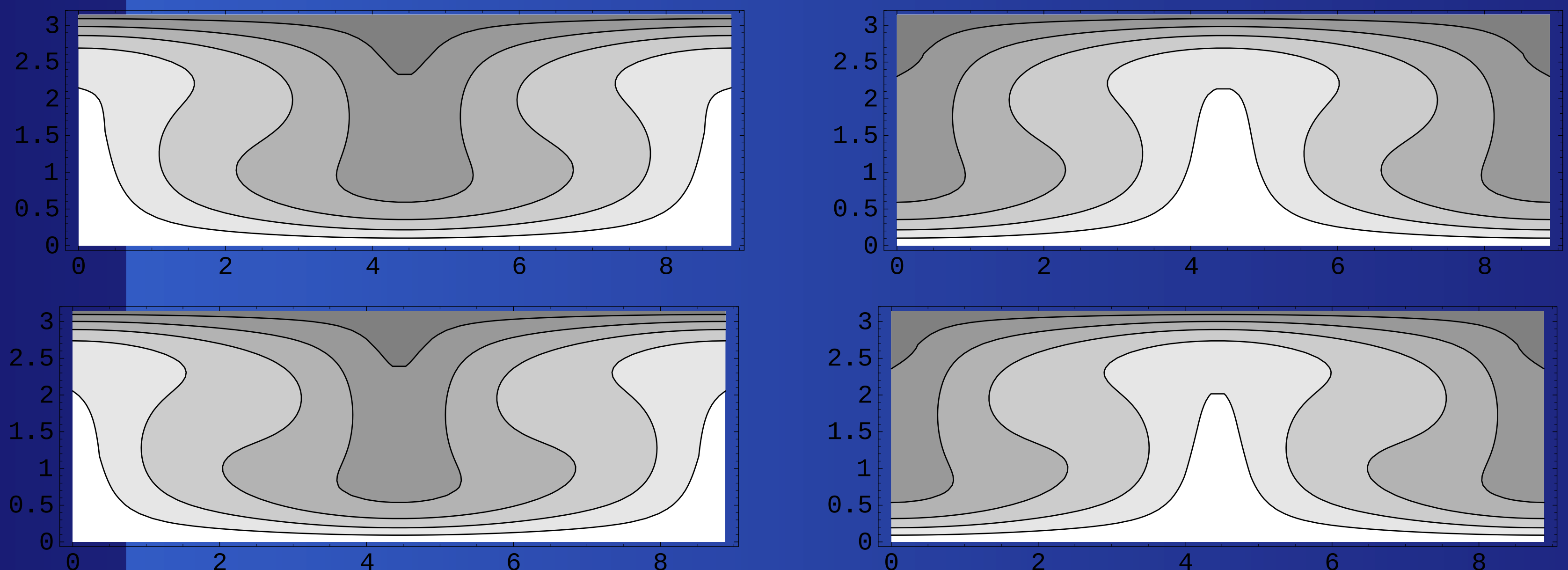

$\mathcal{R} = 30$

$\mathcal{R} = 40$

The First Bifurcated Solutions II

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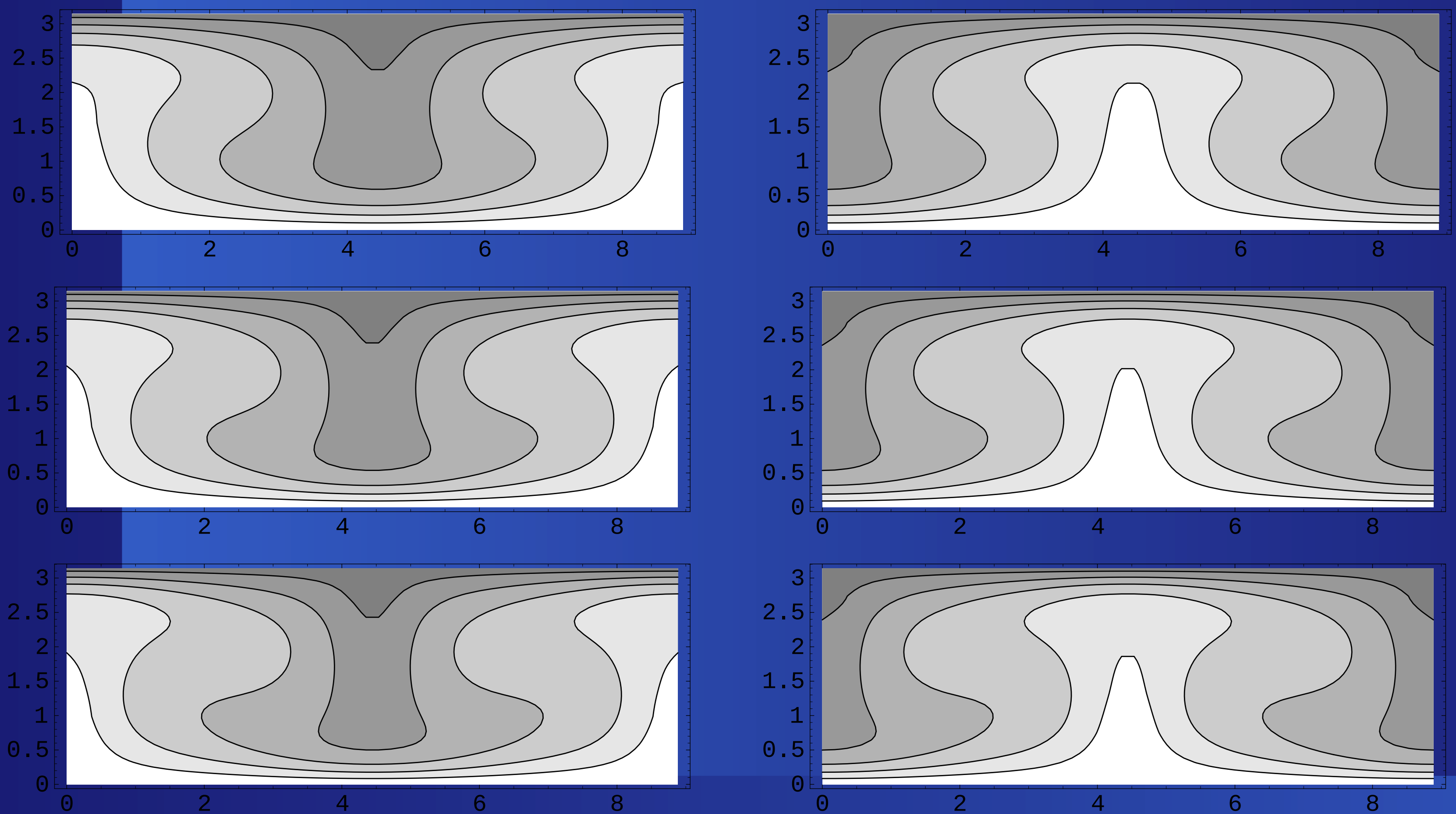

$\mathcal{R} = 30$

$\mathcal{R} = 40$

$\mathcal{R} = 50$

The First Bifurcated Solutions III

The temperature

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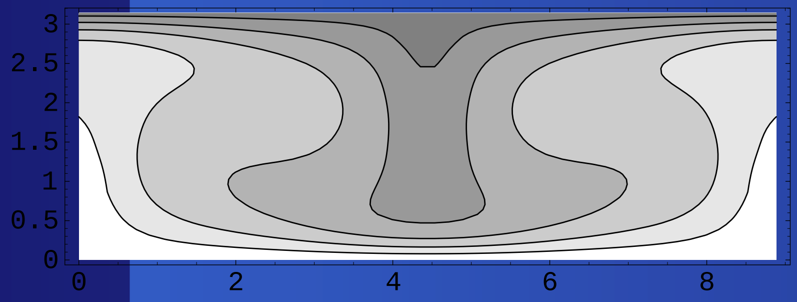

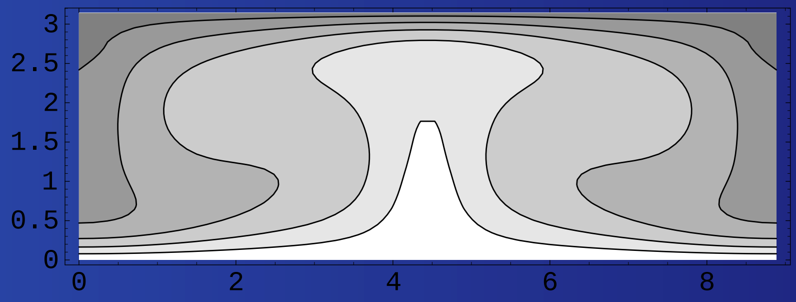

$\mathcal{R} = 60$

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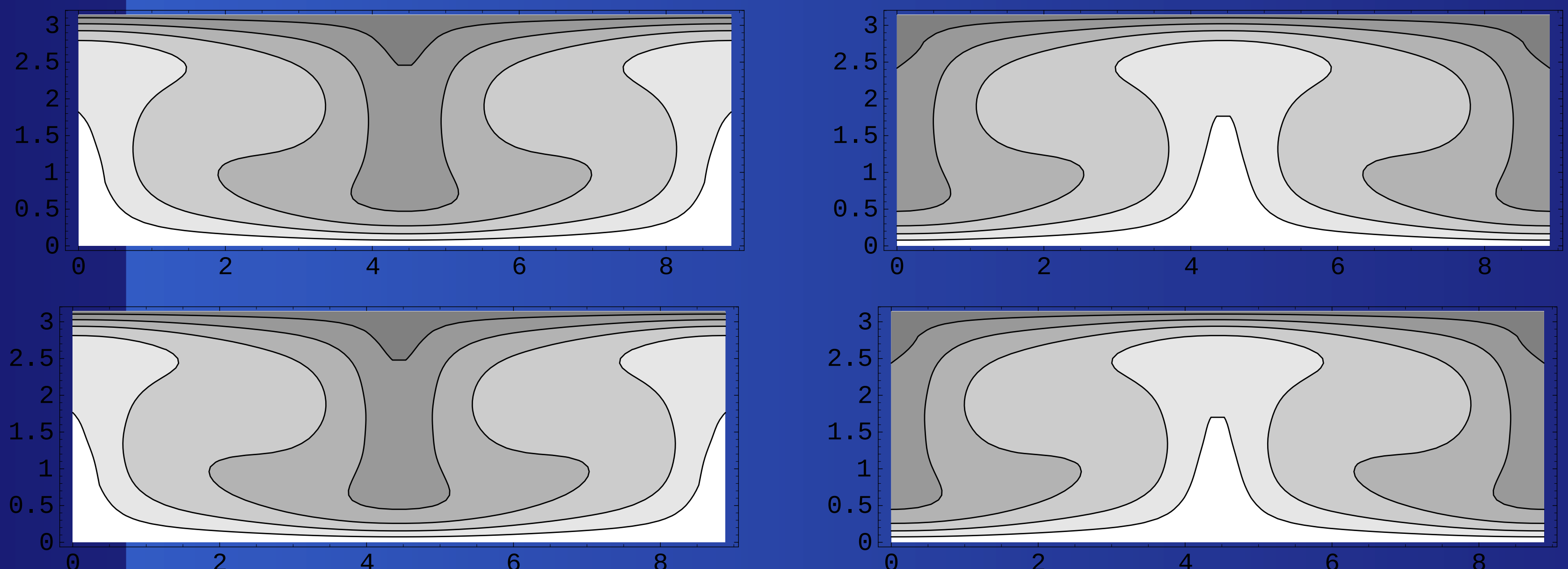

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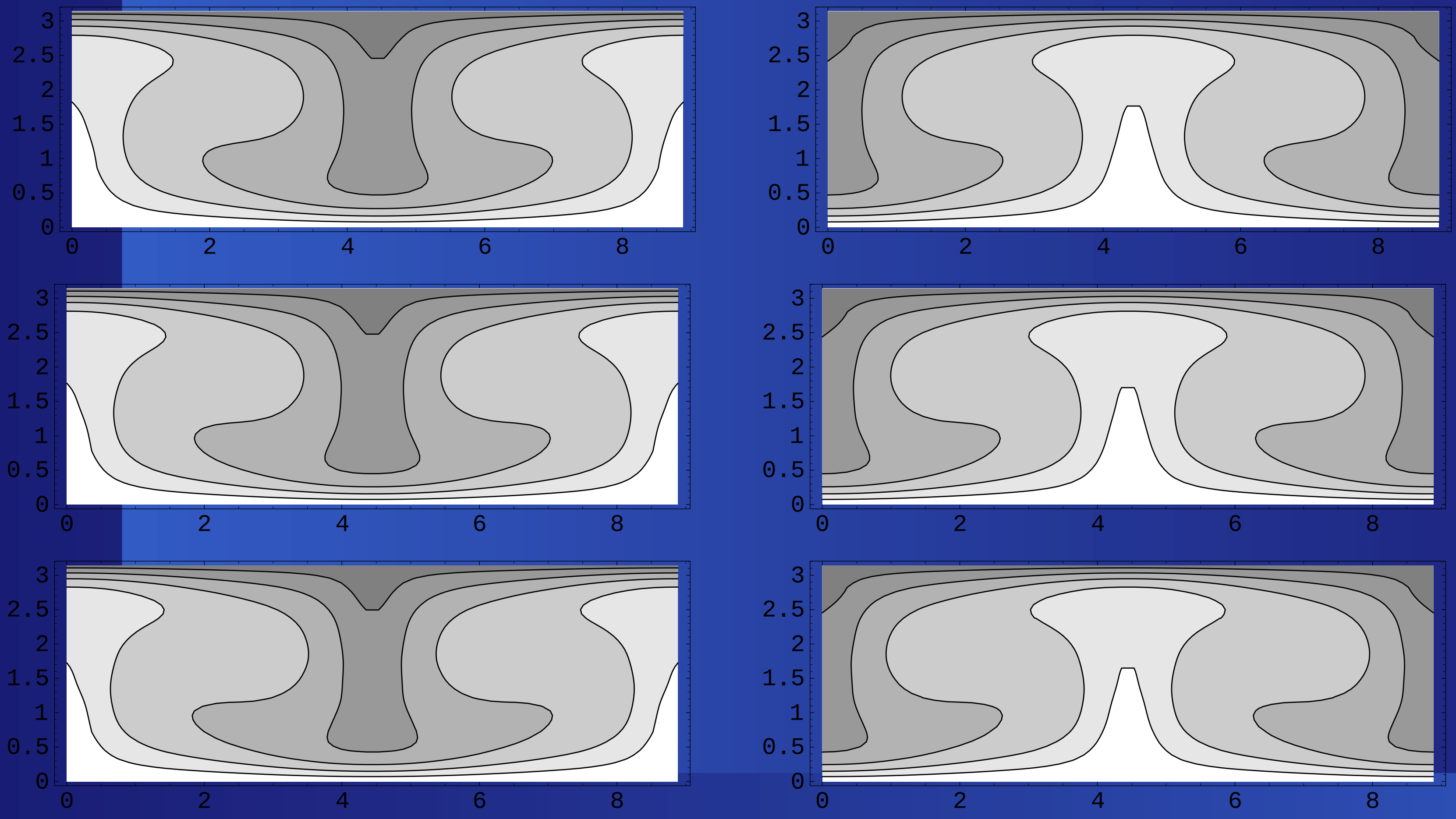

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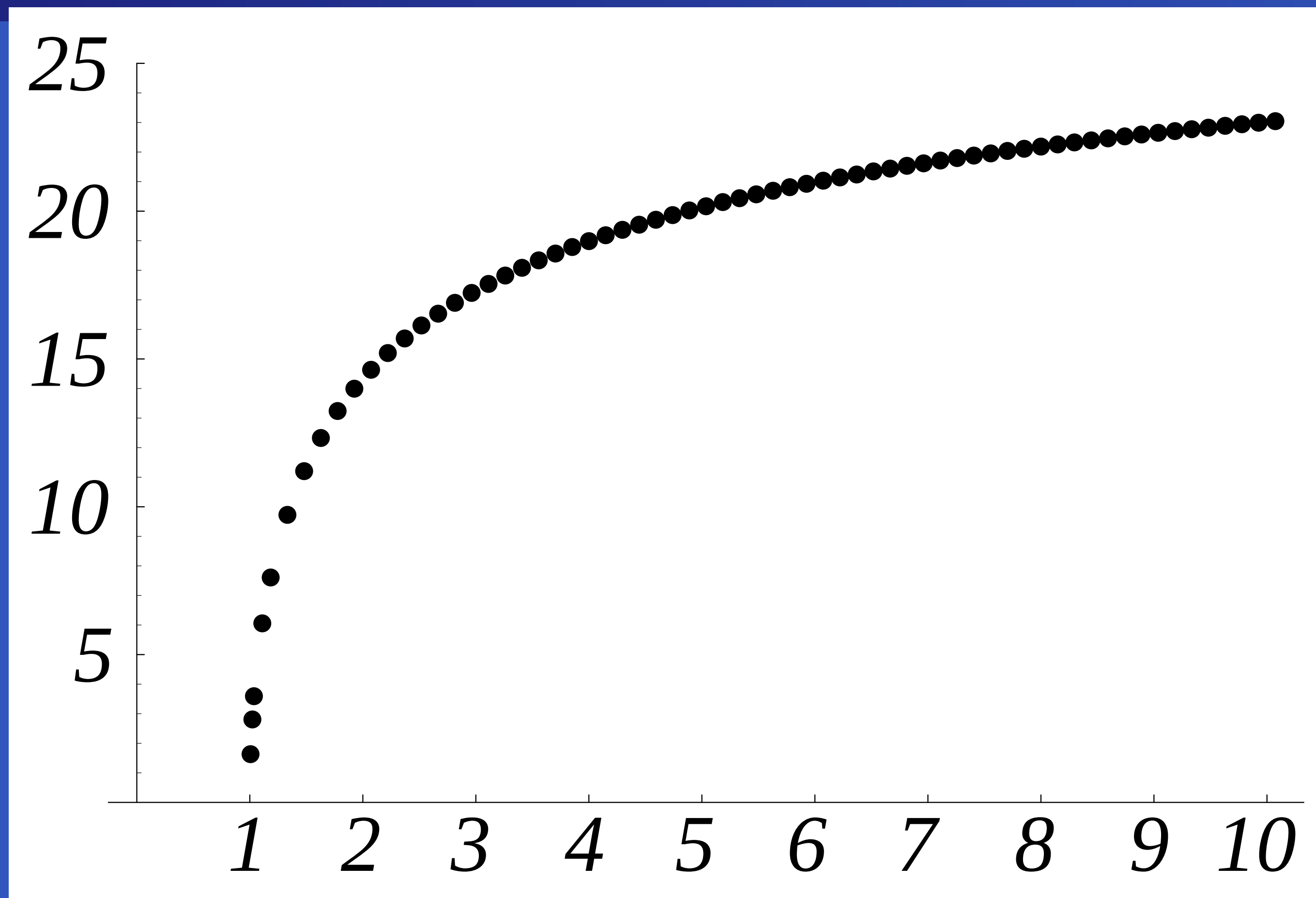
The Bifurcation Diagram I



$$\longrightarrow \mathcal{R}/\mathcal{R}_{\mathcal{C}}$$

The norm is defined by $\max\limits_{m,n}\{|\hat{B}_{mn}|\}$ for $\hat{\Theta}_N = \sum_{m=0}^{M_2}\sum_{n=1}^{N_2}\hat{B}_{mn}\cos(amx)\sin(nz)$.

The Second bifurcation

At a Rayleigh number:

$$\mathcal{R} = \frac{(a^2m^2 + n^2)^3}{a^2m^2} = 13.5 \quad (m = 2, n = 1, a = 1/\sqrt{2}),$$

the second bifurcation from the trivial solution occurs.

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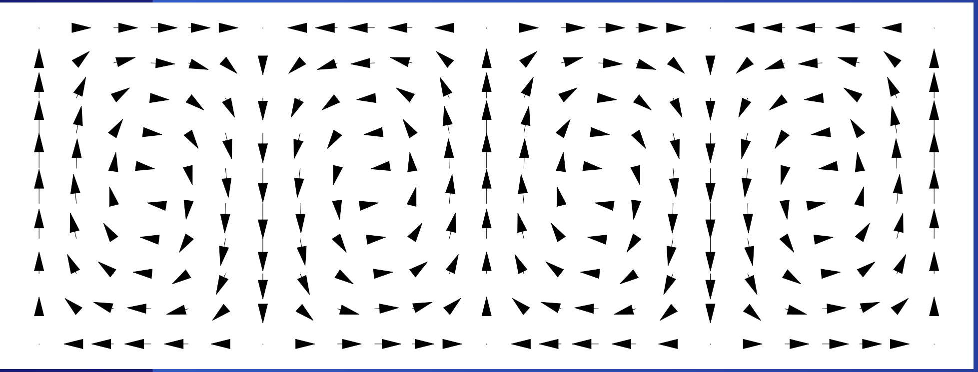
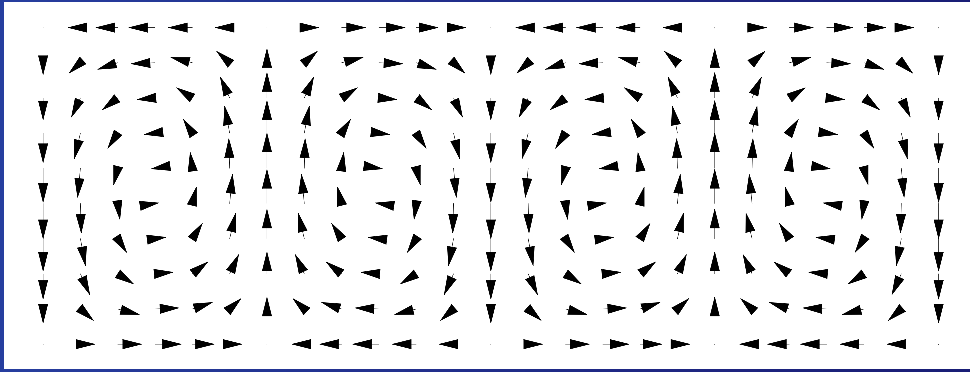
EXAMPLE: The velocity field $(-(\hat{\Psi}_N)_z, (\hat{\Psi}_N)_x)$;

$$\hat{\Psi}_N = \sum_{m=1}^{M_1}\sum_{n=1}^{N_1}\hat{A}_{mn}\sin(amx)\sin(nz),\ \mathcal{P}=10,\ \mathcal{R}=50,\ M_1=N_1=10.$$



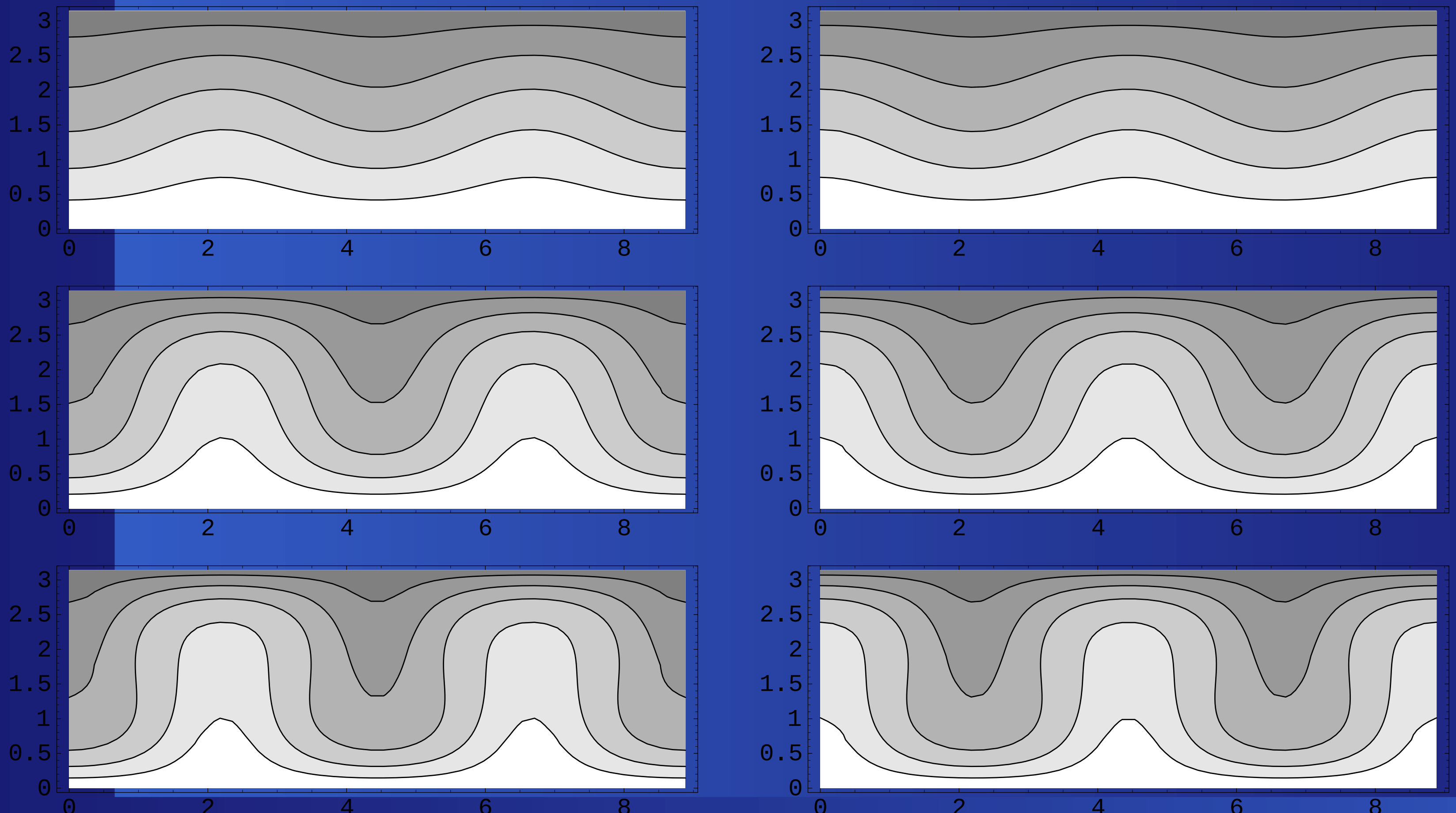


$\hat{A}_{21} \approx -7.026$ $\hat{A}_{21} \approx 7.026$

The Second Bifurcated Solutions I

The temperature

$$\theta = \delta T(1 - z/\pi - \Theta/\sqrt{\mathcal{RP}}\pi) + T, \qquad T = 0,\ \delta T = 5,\ \mathcal{P} = 10$$



$\mathcal{R} = 14$

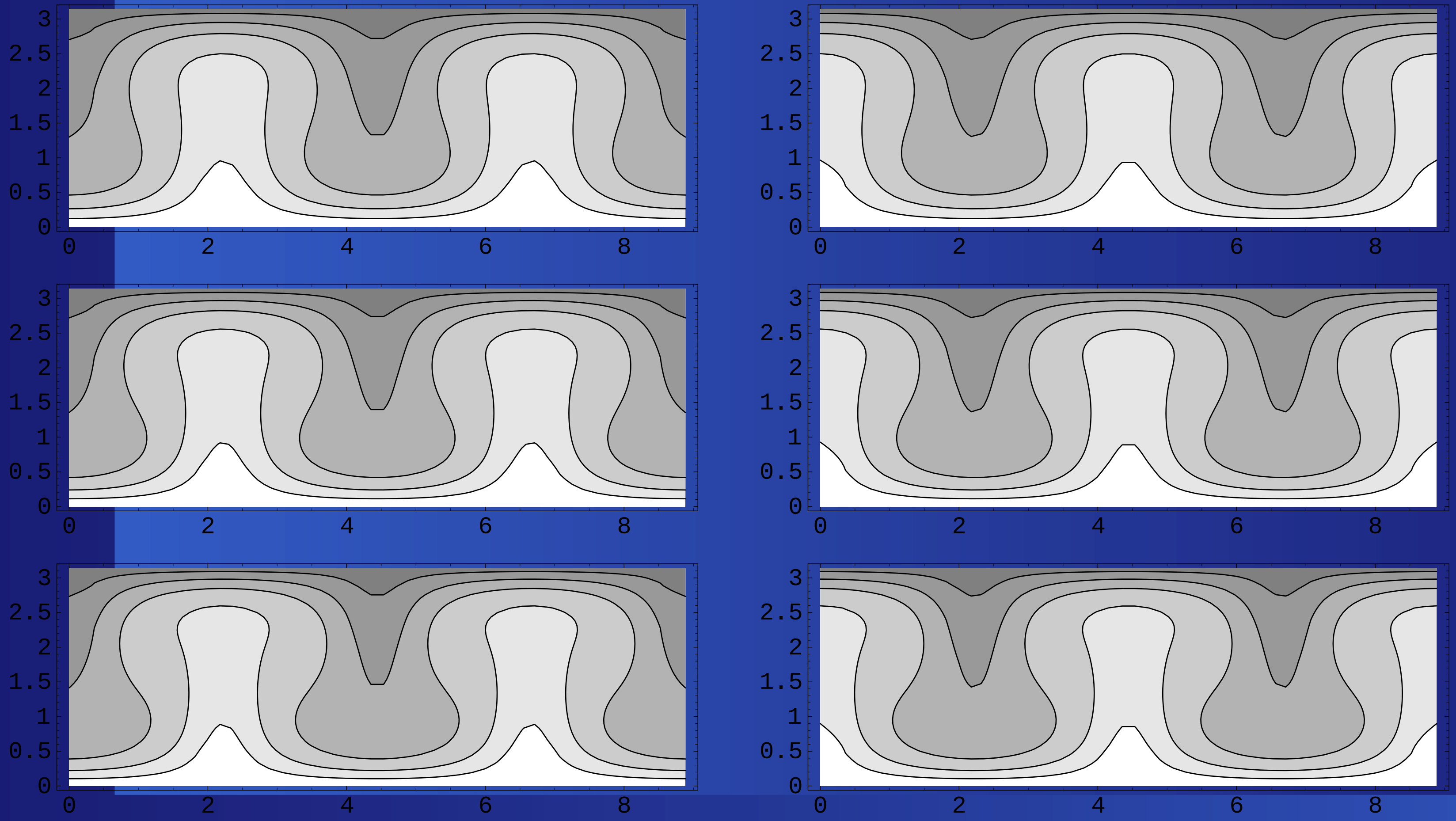
$\mathcal{R} = 20$

$\mathcal{R} = 30$

The Second Bifurcated Solutions II

The temperature

$$\theta = \delta T(1 - z/\pi - \Theta/\sqrt{\mathcal{R}\mathcal{P}}\pi) + T, \qquad T = 0,\ \delta T = 5,\ \mathcal{P} = 10$$



$\mathcal{R} = 40$

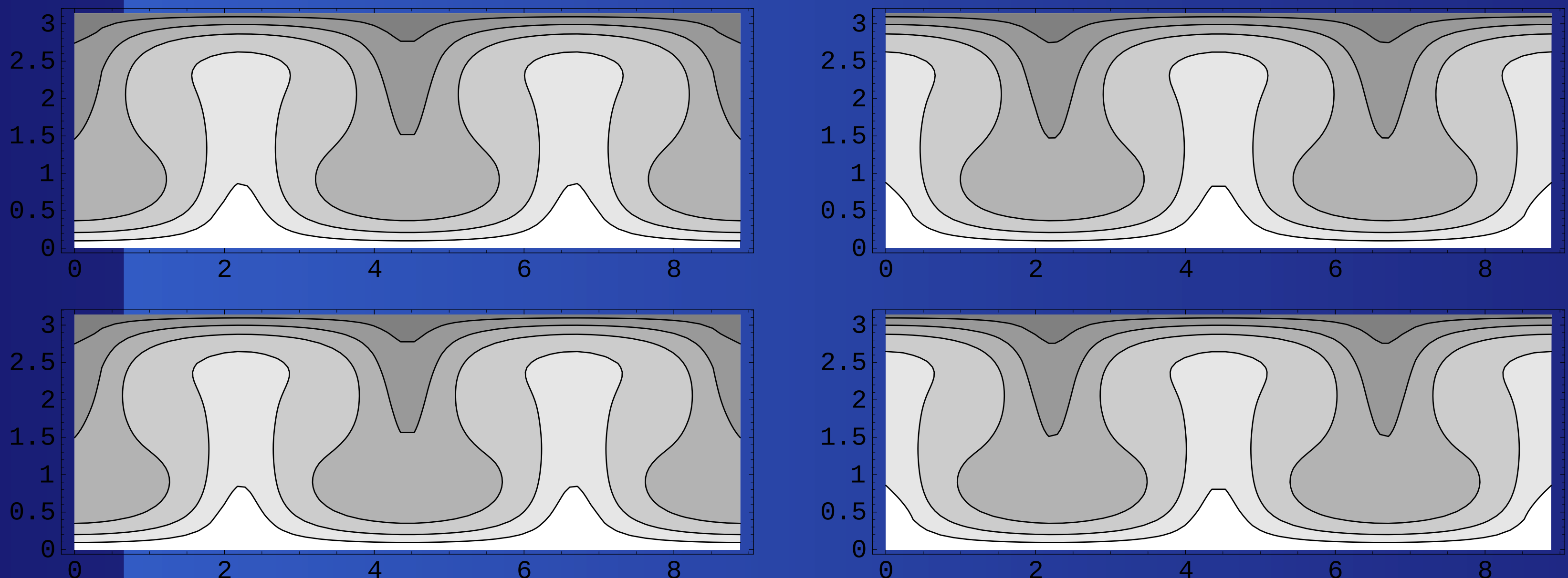
$\mathcal{R} = 50$

$\mathcal{R} = 60$

The Second Bifurcated Solutions III

The temperature

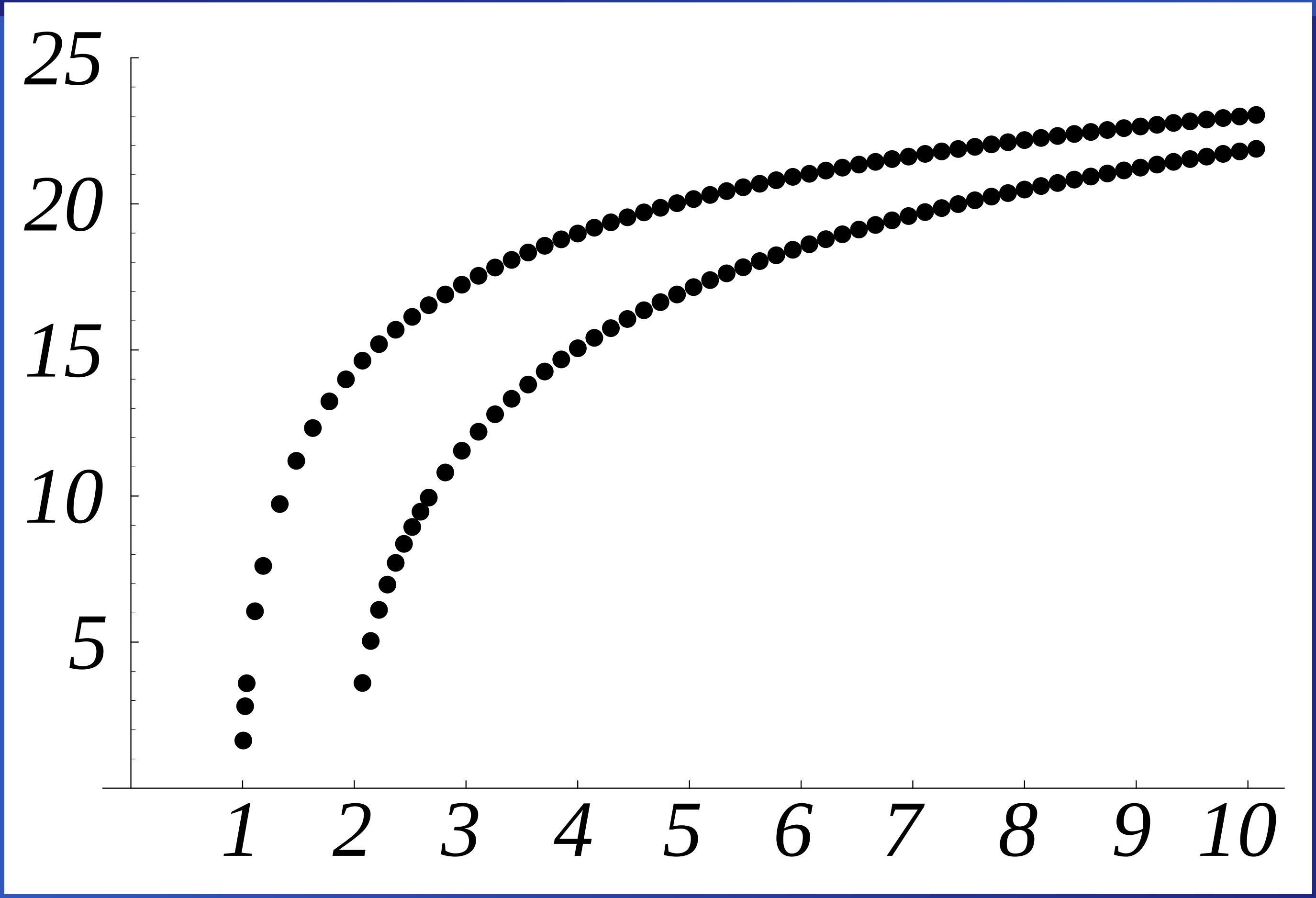
$$\theta = \delta T(1 - z/\pi - \Theta/\sqrt{\mathcal{R}\mathcal{P}}\pi) + T, \qquad T = 0,\ \delta T = 5,\ \mathcal{P} = 10$$



$\mathcal{R} = 70$

$\mathcal{R} = 80$

The Bifurcation Diagram II



$\longrightarrow \mathcal{R}/\mathcal{R}_{\mathcal{C}}$

The norm is defined by $\max_{m,n}\{|\hat{B}_{mn}|\}$ for $\hat{\Theta}_N = \sum_{m=0}^{M_2}\sum_{n=1}^{N_2} \hat{B}_{mn}\cos(amx)\sin(nz)$.

The third bifurcation

At a Rayleigh number:

$$\mathcal{R} = \frac{(a^2m^2 + n^2)^3}{a^2m^2} = 1331/36 \quad (m = 3, n = 1, a = 1/\sqrt{2}),$$

the third bifurcation from the trivial solution occurs.

The third bifurcation

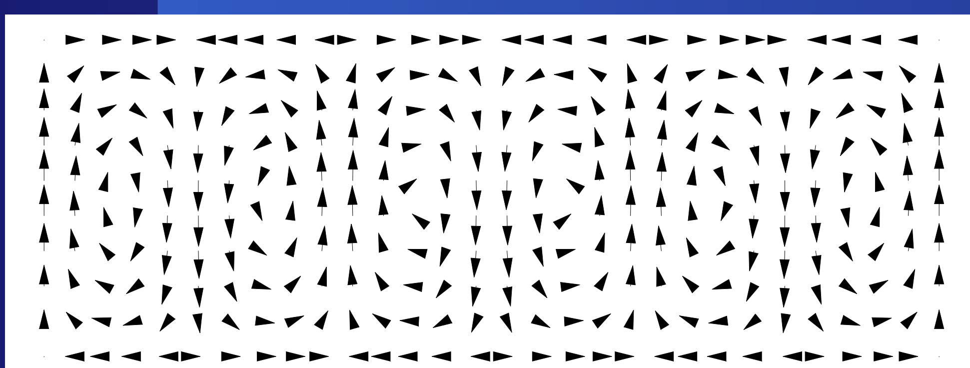
At a Rayleigh number:

$$\mathcal{R} = \frac{(a^2m^2 + n^2)^3}{a^2m^2} = 1331/36 \quad (m = 3, n = 1, a = 1/\sqrt{2}),$$

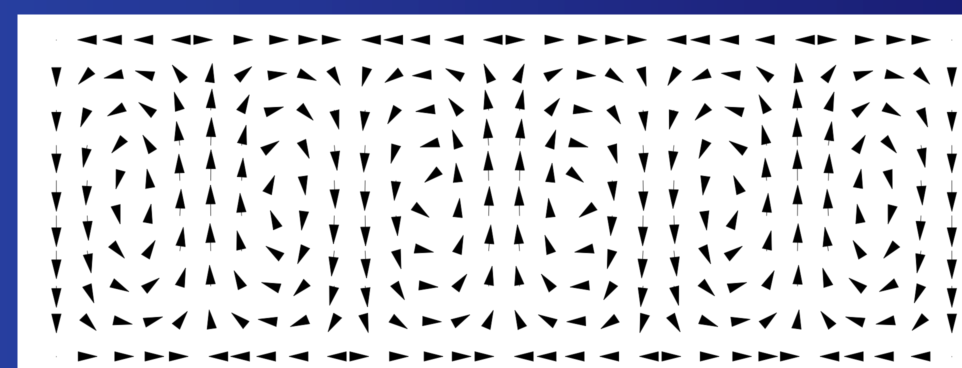
the third bifurcation from the trivial solution occurs.

EXAMPLE: The velocity field $(-(\hat{\Psi}_N)_z, (\hat{\Psi}_N)_x)$;

$$\hat{\Psi}_N = \sum_{m=1}^{M_1} \sum_{n=1}^{N_1} \hat{A}_{mn} \sin(amx) \sin(nz), \ \mathcal{P} = 10, \ \mathcal{R} = 50, \ M_1 = N_1 = 10.$$



$\hat{A}_{31} \approx -2.029$

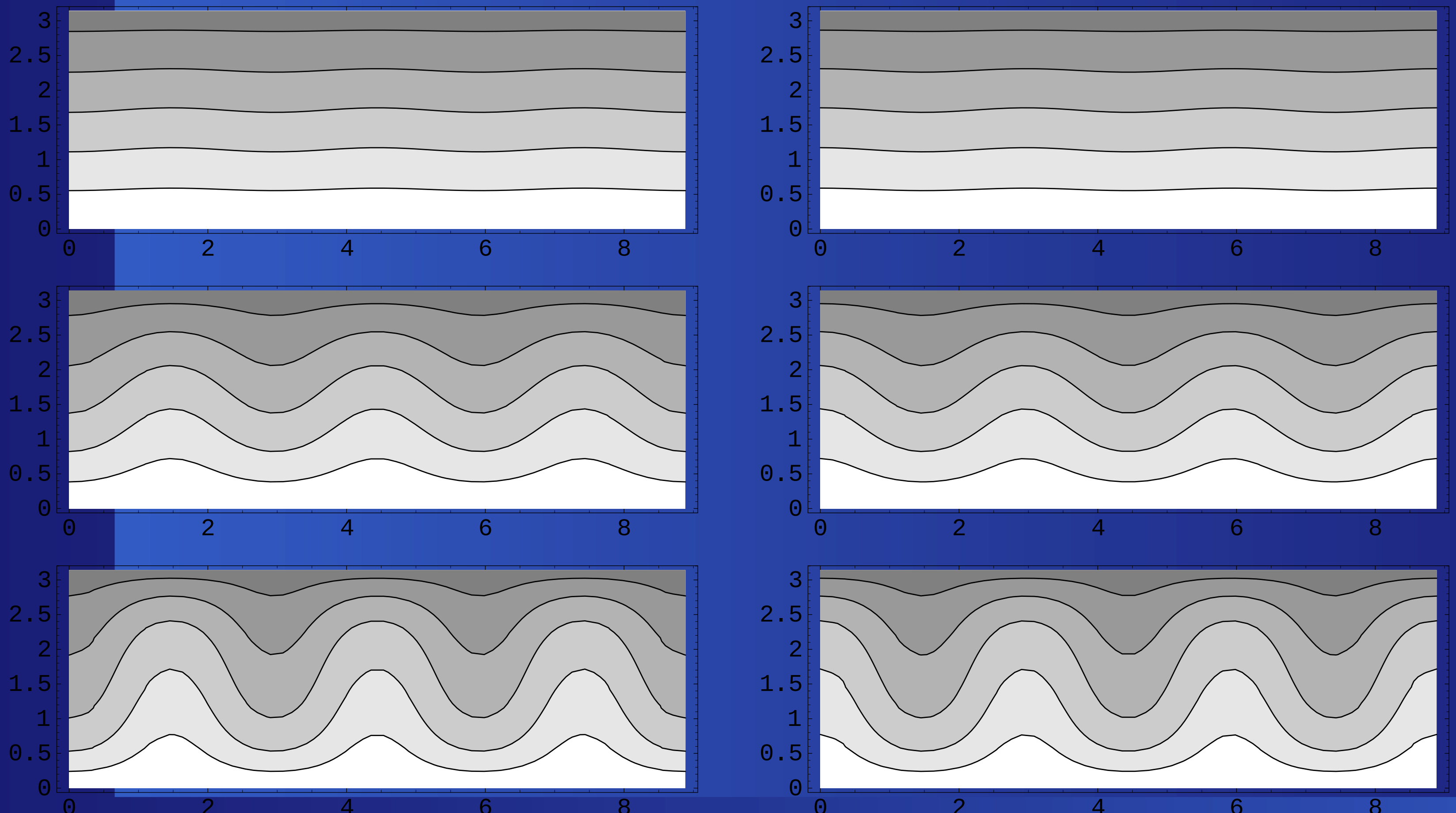


$\hat{A}_{31} \approx 2.029$

The Third Bifurcated Solutions I

The temperature

$$\theta = \delta T(1 - z/\pi - \Theta/\sqrt{\mathcal{RP}}\pi) + T, \qquad T = 0,\ \delta T = 5,\ \mathcal{P} = 10$$



$\mathcal{R} = 37$

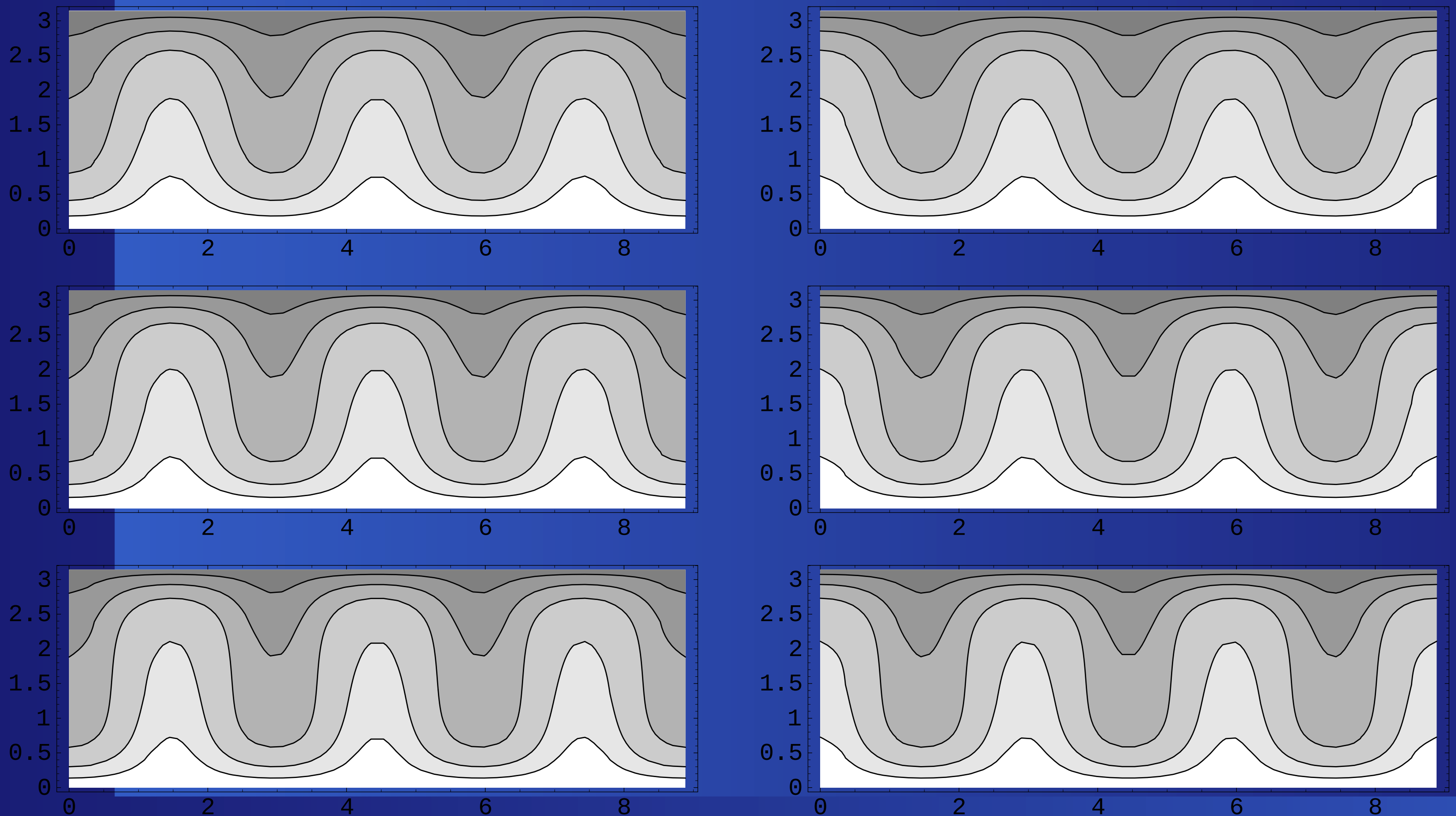
$\mathcal{R} = 40$

$\mathcal{R} = 50$

The Third Bifurcated Solutions II

The temperature

$$\theta = \delta T(1 - z/\pi - \Theta/\sqrt{\mathcal{RP}}\pi) + T, \qquad T = 0,\ \delta T = 5,\ \mathcal{P} = 10$$

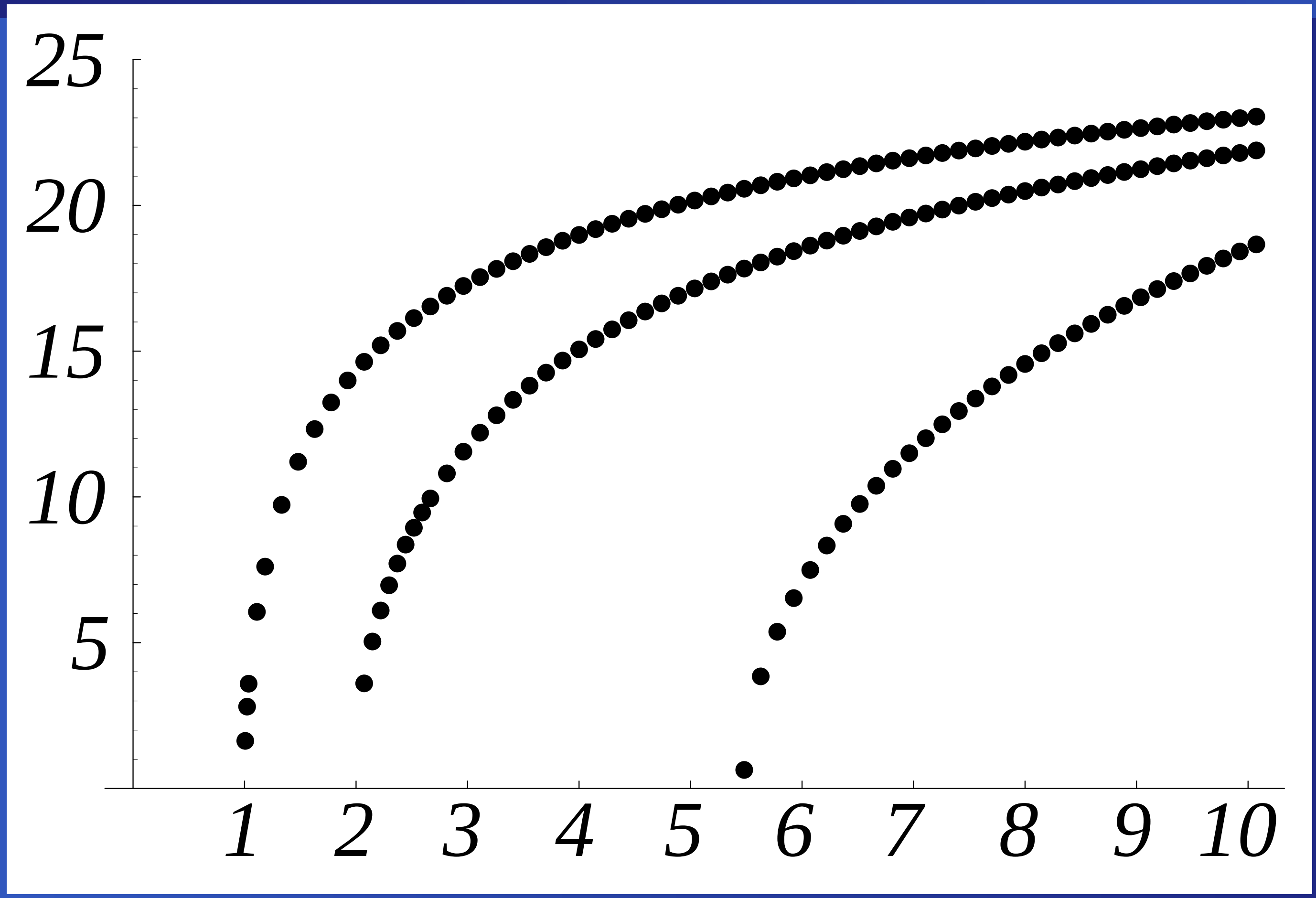


$\mathcal{R} = 60$

$\mathcal{R} = 70$

$\mathcal{R} = 80$

The Bifurcation Diagram III

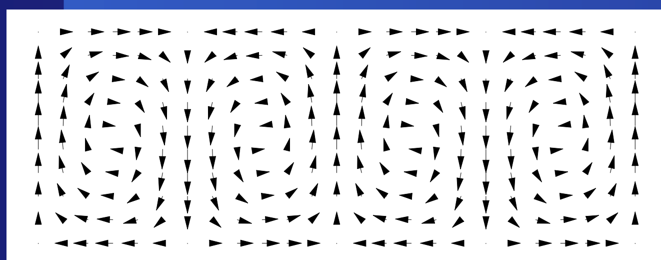


$$\longrightarrow \mathcal{R}/\mathcal{R}_{\mathcal{C}}$$

The norm is defined by $\max_{m,n}\{|\hat{B}_{mn}|\}$ for $\hat{\Theta}_N = \sum_{m=0}^{M_2}\sum_{n=1}^{N_2} \hat{B}_{mn}\cos(amx)\sin(nz)$.

Other bifurcation

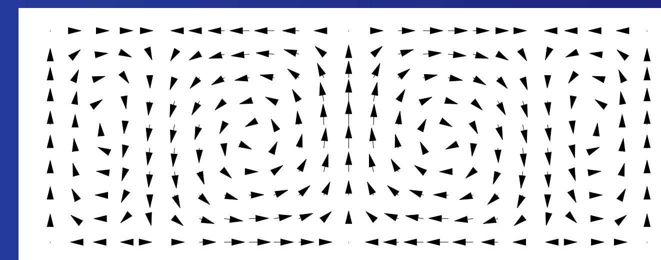
At $\mathcal{R} \sim 32.5$, there exist another four nontrivial solutions:

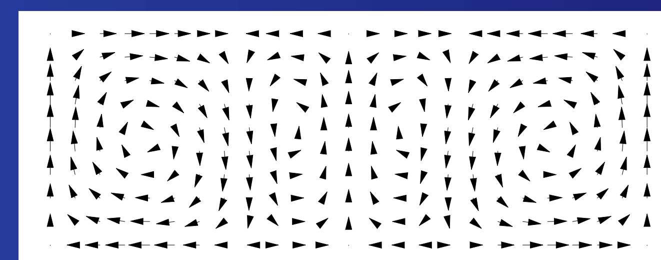


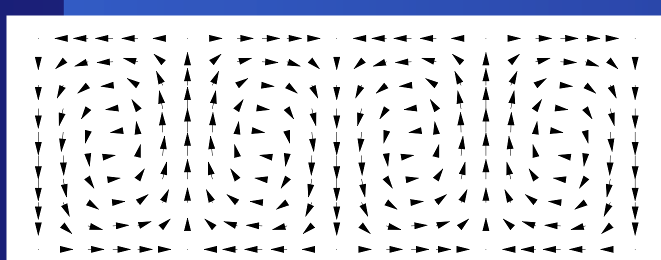
2nd bifurcated solution from trivial solution (1)

? ↗

? ↘



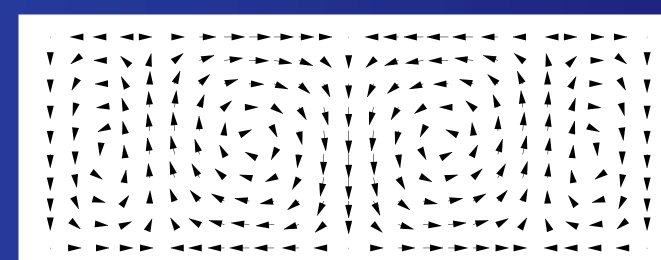


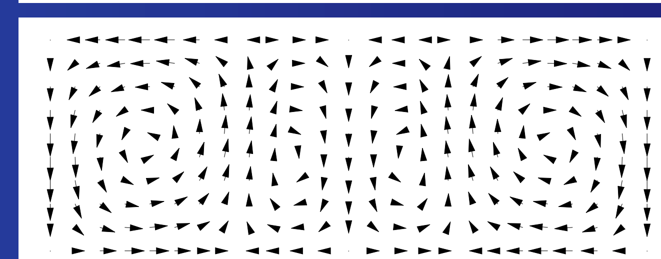


2nd bifurcated solution from trivial solution (2)

? ↗

? ↘

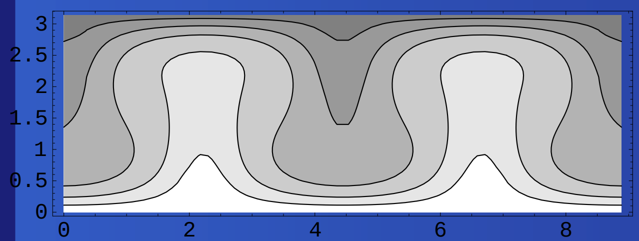




The Other Bifurcated Solutions I

The temperature

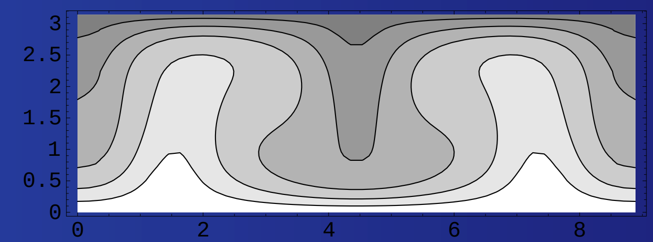
$$\theta = \delta T(1 - z/\pi - \Theta/\sqrt{\mathcal{RP}}\pi) + T, \qquad T = 0,\ \delta T = 5,\ \mathcal{P} = 10, \mathcal{R} = 33$$

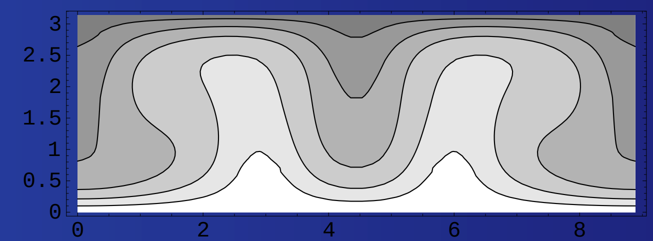


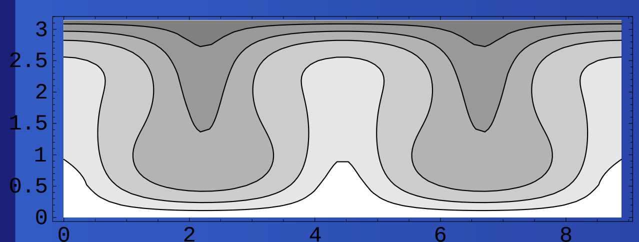
2nd bifurcated solution from trivial solution (1)

? ↗

? ↘



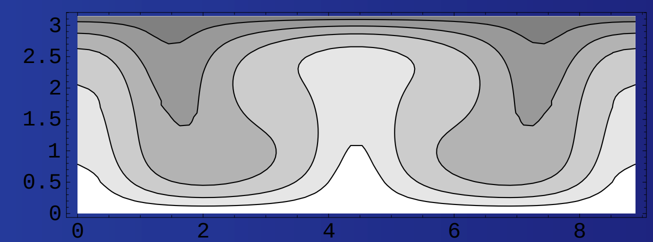


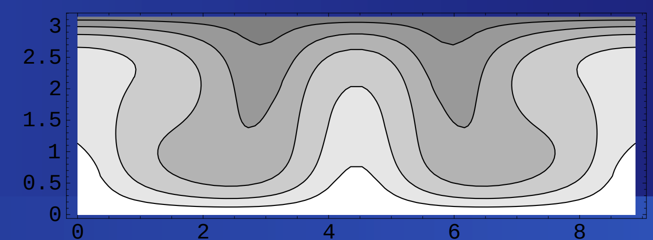


2nd bifurcated solution from trivial solution (2)

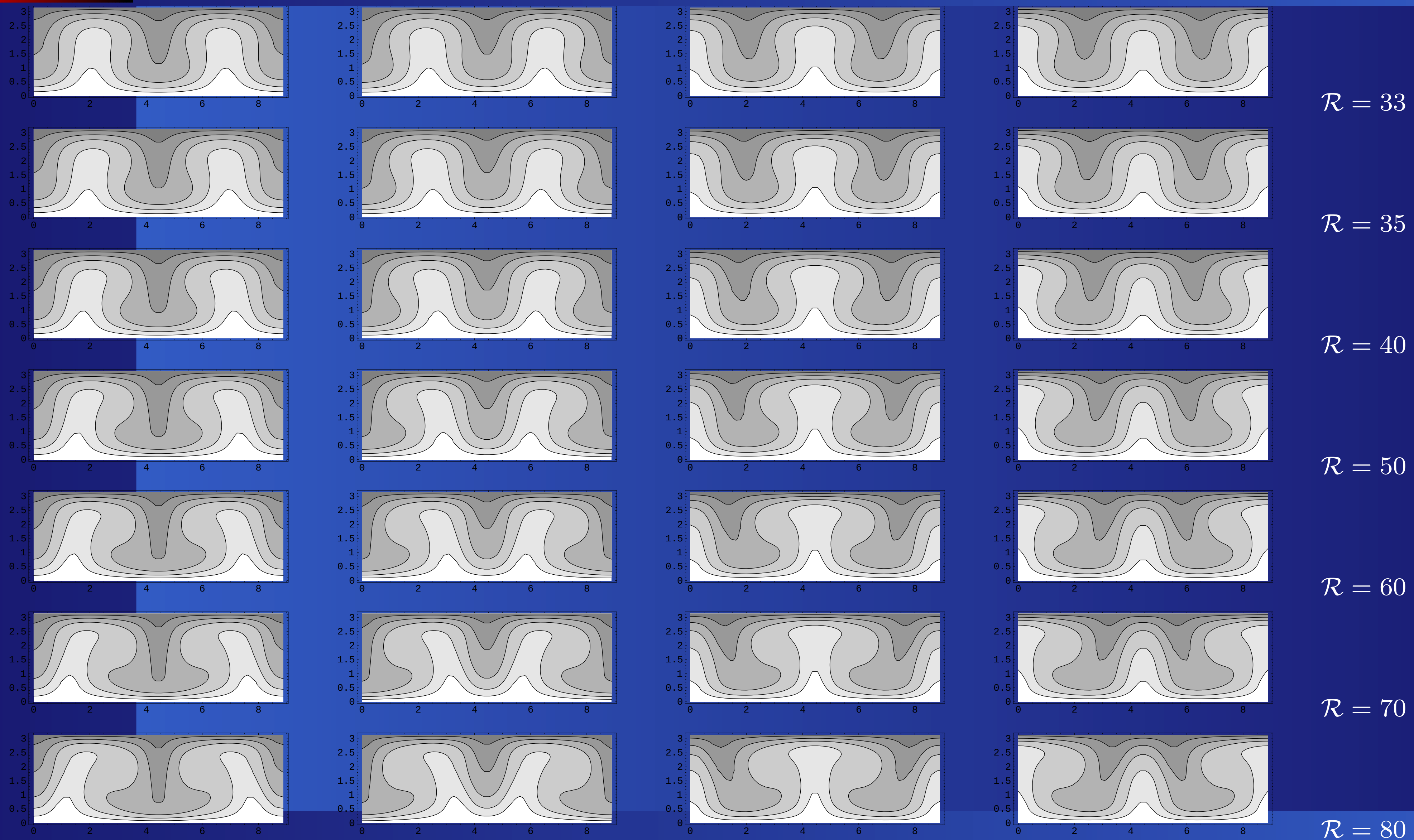
? ↗

? ↘





The Other Bifurcated Solutions II



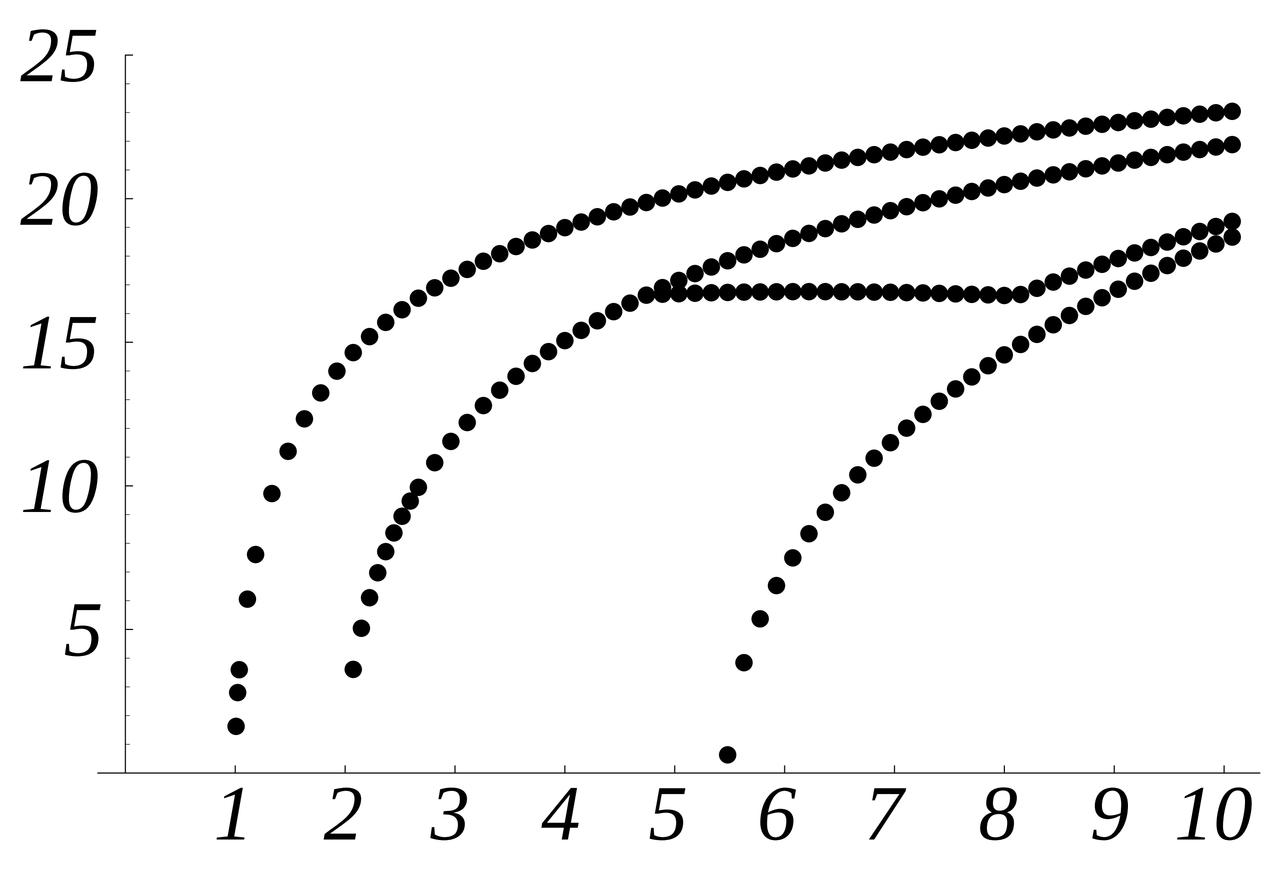
Verification Results

$No.$	N	$\|\hat{\Psi}_N\|_{L^2}$	$\|\hat{\Theta}_N\|_{L^2}$	$\|W_N^{(1)}\|_{L^\infty}$	$\|W_N^{(2)}\|_{L^\infty}$	$\|W_*^{(1)}\|_{L^\infty}$	$\|W_*^{(2)}\|_{L^\infty}$
1	45	17.44	34.89	1.40×10^{-9}	3.12×10^{-11}	2.46×10^{-11}	1.26×10^{-7}
2	45	17.44	34.89	1.40×10^{-9}	3.12×10^{-11}	2.46×10^{-11}	1.26×10^{-7}
3	30	8.14	30.57	2.35×10^{-6}	2.56×10^{-8}	7.75×10^{-8}	1.35×10^{-4}
4	30	8.14	30.57	2.35×10^{-6}	2.56×10^{-8}	7.75×10^{-8}	1.35×10^{-4}
5	50	9.62	29.43	9.75×10^{-9}	8.77×10^{-10}	6.96×10^{-11}	5.21×10^{-7}
6	50	9.62	29.43	9.75×10^{-9}	8.77×10^{-10}	6.96×10^{-11}	5.21×10^{-7}
7	50	9.62	29.43	9.75×10^{-9}	8.77×10^{-10}	6.96×10^{-11}	5.21×10^{-7}
8	50	9.62	29.43	9.75×10^{-9}	8.77×10^{-10}	6.96×10^{-11}	5.21×10^{-7}
9	20	2.84	19.49	3.40×10^{-5}	9.56×10^{-7}	1.75×10^{-6}	1.10×10^{-3}
10	20	2.84	19.49	3.40×10^{-5}	9.56×10^{-7}	1.75×10^{-6}	1.10×10^{-3}

$\mathcal{R} = 60$, $\mathcal{P} = 10$; $N := M_1 = M_2 = N_1 = N_2$; matrix dimension $= N(2N+1)$.
For each $\mathcal{R}$, there exists a solution $(\Psi, \Theta) \in X^3 \times Y^1$ in the candidate set:

$$\begin{aligned}\Psi &\in \hat{\Psi}_N + W_N^{(1)} + W_*^{(1)},\\ \Theta &\in \hat{\Theta}_N + W_N^{(2)} + W_*^{(2)}.\end{aligned}$$

Verified Bifurcated Solutions



$\longrightarrow \mathcal{R}/\mathcal{R}_{\mathcal{C}}$

The norm is defined by $\max_{m,n}\{|\hat{B}_{mn}|\}$ for $\hat{\Theta}_N = \sum_{m=0}^{M_2}\sum_{n=1}^{N_2}\hat{B}_{mn}\cos(amx)\sin(nz)$.

On Slides

pLaTeX2e (based on LaTeX2e)

- prosper.cls (http://prosper.sourceforge.net/)
- seminar.cls

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