Proof of Lemma 4.2

Lemma 4.2

For $N = 200$, the operator $q : X \to Y$ is bounded and upper bounds $\nu_k > 0$ $(k = 1, 2, 3)$ satisfying

$$\|P_h \mathscr{A}^{-1} q w_*\|_X \le \nu_1 \|w_*\|_X, \qquad \forall w_* \in (I - P_h)X, \tag{1}$$
$$\|qw\|_Y \le \nu_2 \|P_h w\|_X + \nu_3 \|(I - P_h)w\|_X, \qquad \forall w \in X \tag{2}$$

can be taken as

$$\nu_1 = 0.5762, \tag{3}$$
$$\nu_2 = 6467, \tag{4}$$
$$\nu_3 = 12750, \tag{5}$$

respectively.

1 Bound of ν_1

Each $w_* \in (I - P_h)X$ can be represented by $w_* = [u_r^\perp, u_i^\perp, v_r^\perp, v_i^\perp, 0, 0, 0, 0]^T$ for $u_r^\perp, u_i^\perp, v_r^\perp, v_i^\perp \in S_h^\perp$, where $S_h^\perp$ stands for the orthogonal complement of S_h such that $H_0^2(\Omega) = S_h \oplus S_h^\perp$. From the definition of $q : X \to Y$ and $f'[0]w_* : X \to (L^2(\Omega))^4$, noting that $u_r^0, u_i^0, v_r^0, v_i^0 \in S_h$ (in the definition of q), we have

$$qw_* = \begin{bmatrix} \hat{q}_1 u_r^\perp - \hat{q}_2 u_i^\perp \\ \hat{q}_2 u_r^\perp + \hat{q}_1 u_i^\perp \\ \hat{q}_4 u_r^\perp - \hat{q}_3 u_i^\perp + \hat{q}_1 v_r^\perp - \hat{q}_2 v_i^\perp \\ \hat{q}_3 u_r^\perp + \hat{q}_4 u_i^\perp + \hat{q}_2 v_r^\perp + \hat{q}_1 v_i^\perp \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{6}$$

By the Lax & Milgram lemma, for any $g \in L^2(\Omega)$ there exists a unique solution $\omega \in H^4(\Omega) \cap H_0^2(\Omega)$ satisfying $D^4\omega = g$ which we denote by $\omega = \mathcal{K}g$. Then using $\mathcal{K} : L^2(\Omega) \to H^4(\Omega) \cap H_0^2(\Omega)$, we have

$$\mathscr{A}^{-1} q w_* = \begin{bmatrix} \mathcal{K}(\hat{q}_1 u_r^\perp - \hat{q}_2 u_i^\perp) \\ \mathcal{K}(\hat{q}_2 u_r^\perp + \hat{q}_1 u_i^\perp) \\ \mathcal{K}(\hat{q}_4 u_r^\perp - \hat{q}_3 u_i^\perp + \hat{q}_1 v_r^\perp - \hat{q}_2 v_i^\perp) \\ \mathcal{K}(\hat{q}_3 u_r^\perp + \hat{q}_4 u_i^\perp + \hat{q}_2 v_r^\perp + \hat{q}_1 v_i^\perp) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

then

$$\begin{aligned}
\|P_h \mathscr{A}^{-1} q w_*\|_X^2 = {} & \|P_N \mathcal{K}(\hat{q}_1 u_r^\perp - \hat{q}_2 u_i^\perp)\|_{H_0^2}^2 + \|P_N \mathcal{K}(\hat{q}_2 u_r^\perp + \hat{q}_1 u_i^\perp)\|_{H_0^2}^2 \\
& + \|P_N \mathcal{K}(\hat{q}_4 u_r^\perp - \hat{q}_3 u_i^\perp + \hat{q}_1 v_r^\perp - \hat{q}_2 v_i^\perp)\|_{H_0^2}^2 + \|P_N \mathcal{K}(\hat{q}_3 u_r^\perp + \hat{q}_4 u_i^\perp + \hat{q}_2 v_r^\perp + \hat{q}_1 v_i^\perp)\|_{H_0^2}^2 \\
\le {} & \left(\left\|\mathcal{K}\hat{q}_1 u_r^\perp\right\|_{H_0^2} + \left\|\mathcal{K}\hat{q}_2 u_i^\perp\right\|_{H_0^2} \right)^2 + \left(\left\|\mathcal{K}\hat{q}_2 u_r^\perp\right\|_{H_0^2} + \left\|\mathcal{K}\hat{q}_1 u_i^\perp\right\|_{H_0^2} \right)^2 \\
& + \left(\left\|\mathcal{K}\hat{q}_4 u_r^\perp\right\|_{H_0^2} + \left\|\mathcal{K}\hat{q}_3 u_i^\perp\right\|_{H_0^2} + \left\|\mathcal{K}\hat{q}_1 v_r^\perp\right\|_{H_0^2} + \left\|\mathcal{K}\hat{q}_2 v_i^\perp\right\|_{H_0^2} \right)^2 \\
& + \left(\left\|\mathcal{K}\hat{q}_3 u_r^\perp\right\|_{H_0^2} + \left\|\mathcal{K}\hat{q}_4 u_i^\perp\right\|_{H_0^2} + \left\|\mathcal{K}\hat{q}_2 v_r^\perp\right\|_{H_0^2} + \left\|\mathcal{K}\hat{q}_1 v_i^\perp\right\|_{H_0^2} \right)^2.
\end{aligned}$$

For $u \in \{u_r^\perp, u_i^\perp, v_r^\perp, v_i^\perp\}$, from the definition of $\hat{q}_i$ $(i = 1, \ldots, 4)$ we have

$$\hat{q}_1 u = 2\hat{a}^2 D^2 u - \hat{a}^4 u,$$

$$\begin{aligned}\hat{q}_2 u &= (\hat{a}\hat{R} - \hat{\lambda}_i)D^2 u - \hat{a}\hat{R}x^2 D^2 u + \hat{a}(-\hat{R}(\hat{a}^2 - 2) + \hat{a}\hat{\lambda}_i)u + \hat{a}^3\hat{R}x^2 u,\\ &= \left(\hat{a}\hat{R} - \hat{\lambda}_i - \hat{a}\hat{R}x^2\right) D^2 u + \left(\hat{a}(-\hat{R}(\hat{a}^2 - 2) + \hat{a}\hat{\lambda}_i) + \hat{a}^3\hat{R}x^2\right) u,\end{aligned}$$

$$\begin{aligned}\hat{q}_3 u &= (\hat{R} - \hat{\mu}_i)D^2 u - \hat{R}x^2 D^2 u + (-\hat{R}(3\hat{a}^2 - 2) + \hat{a}(\hat{a}\hat{\mu}_i + 2\hat{\lambda}_i))u + 3\hat{a}^2\hat{R}x^2 u,\\ &= \left(\hat{R} - \hat{\mu}_i - \hat{R}x^2\right) D^2 u + \left(-\hat{R}(3\hat{a}^2 - 2) + \hat{a}(\hat{a}\hat{\mu}_i + 2\hat{\lambda}_i) + 3\hat{a}^2\hat{R}x^2\right) u,\end{aligned}$$

$$\hat{q}_4 u = 4\hat{a}D^2 u - 4\hat{a}^3 u.$$

We note that for each $v \in L^2(\Omega)$, $\psi := \mathcal{K}v$ satisfies $D^4\psi = v$, and

$$\|\mathcal{K}v\|_{H_0^2}^2 = \|\psi\|_{H_0^2}^2 = \left(D^2\psi, D^2\psi\right)_{L^2} = \left(\psi, D^4\psi\right)_{L^2} \le \|\psi\|\,\|v\| = C_p\|\psi\|_{H_0^2}\|v\|$$

then

$$\|\mathcal{K}v\|_{H_0^2} \le C_p\|v\|. \tag{7}$$

Moreover, for each $v \in H_0^2(\Omega)$ and $w \in C^2(\Omega)$, setting (again) $\psi := \mathcal{K}wD^2 v$, we have

$$\begin{aligned}\|\mathcal{K}wD^2 v\|_{H_0^2}^2 &= \|\psi\|_{H_0^2}^2 = \left(D^2\psi, D^2\psi\right)_{L^2} = \left(\psi, D^4\psi\right)_{L^2} = \left(\psi, wD^2 v\right)_{L^2} = \left(\psi w, D^2 v\right)_{L^2}\\ &= \left(-D\psi w - \psi Dw, Dv\right)_{L^2} = \left(D^2\psi w + D\psi Dw + D\psi Dw + \psi D^2 w, v\right)_{L^2}\\ &= \left(D^2\psi w + 2D\psi Dw + \psi D^2 w, v\right)_{L^2}\\ &\le \left(\left\|D^2\psi\right\|\|w\|_\infty + 2\|D\psi\|\,\|Dw\|_\infty + \|\psi\|\left\|D^2 w\right\|_\infty\right)\|v\|\\ &\le \left(\|w\|_\infty + 2\sqrt{C_p}\|Dw\|_\infty + C_p\left\|D^2 w\right\|_\infty\right)\|\psi\|_{H_0^2}\|v\|, \qquad \longleftarrow \|w'\| \le \frac{2}{\pi}\|w''\| \text{ by Rayleigh–Ritz}\end{aligned}$$

then

$$\left\|\mathcal{K}wD^2 v\right\|_{H_0^2} \le \left(\|w\|_\infty + 2\sqrt{C_p}\|Dw\|_\infty + C_p\left\|D^2 w\right\|_\infty\right)\|v\|. \tag{8}$$

Hence using (7) and (8) we have

$$\begin{aligned}\|\mathcal{K}\hat{q}_1 u\|_{H_0^2} &\le 2\hat{a}^2\left\|\mathcal{K}D^2 u\right\|_{H_0^2} + \hat{a}^4\|\mathcal{K}u\|_{H_0^2}\\ &\le 2\hat{a}^2\|u\| + \hat{a}^4 C_p\|u\|\\ &= \hat{a}^2(2 + \hat{a}^2 C_p)\|u\|,\end{aligned}$$

$$\begin{aligned}\|\mathcal{K}\hat{q}_2 u\|_{H_0^2} &\le \left\|\mathcal{K}\left(\hat{a}\hat{R} - \hat{\lambda}_i - \hat{a}\hat{R}x^2\right)D^2 u\right\|_{H_0^2} + \left\|\mathcal{K}\left(\hat{a}(-\hat{R}(\hat{a}^2 - 2) + \hat{a}\hat{\lambda}_i) + \hat{a}^3\hat{R}x^2\right)u\right\|_{H_0^2}\\ &\le \left(\left\|\hat{a}\hat{R} - \hat{\lambda}_i - \hat{a}\hat{R}x^2\right\|_\infty + 2\sqrt{C_p}\left\|(\hat{a}\hat{R} - \hat{\lambda}_i - \hat{a}\hat{R}x^2)'\right\|_\infty + C_p\left\|(\hat{a}\hat{R} - \hat{\lambda}_i - \hat{a}\hat{R}x^2)''\right\|_\infty\right)\|u\|\\ &\quad + C_p\left\|\hat{a}(-\hat{R}(\hat{a}^2 - 2) + \hat{a}\hat{\lambda}_i) + \hat{a}^3\hat{R}x^2\right\|_\infty\|u\|\\ &= \left(\left\|\hat{a}\hat{R} - \hat{\lambda}_i - \hat{a}\hat{R}x^2\right\|_\infty + 2\sqrt{C_p}\left\|-2\hat{a}\hat{R}x\right\|_\infty + C_p\left\|-2\hat{a}\hat{R}\right\|_\infty + C_p\left\|\hat{a}(-\hat{R}(\hat{a}^2 - 2) + \hat{a}\hat{\lambda}_i) + \hat{a}^3\hat{R}x^2\right\|_\infty\right)\|u\|\\ &= \left(\left\|\hat{a}\hat{R} - \hat{\lambda}_i - \hat{a}\hat{R}x^2\right\|_\infty + 4\sqrt{C_p}\hat{a}\hat{R} + 2C_p\hat{a}\hat{R} + C_p\left\|\hat{a}(-\hat{R}(\hat{a}^2 - 2) + \hat{a}\hat{\lambda}_i) + \hat{a}^3\hat{R}x^2\right\|_\infty\right)\|u\|.\end{aligned}$$

Here noting that $\Omega = [-1, 1]$, $\hat{\lambda}_i \approx 1555$, $\hat{a} \approx 1.020$, and $\hat{R} \approx 5772$, it is true that

$$\left\|\hat{a}\hat{R} - \hat{\lambda}_i - \hat{a}\hat{R}x^2\right\|_\infty = \max\{\hat{\lambda}_i, \hat{a}\hat{R} - \hat{\lambda}_i\} = \hat{a}\hat{R} - \hat{\lambda}_i,$$

$$\left\|\hat{a}(-\hat{R}(\hat{a}^2 - 2) + \hat{a}\hat{\lambda}_i) + \hat{a}^3\hat{R}x^2\right\|_\infty = \max\{-\hat{a}^3\hat{R} + 2\hat{R}\hat{a} + \hat{a}^2\hat{\lambda}_i,\ 2\hat{R}\hat{a} + \hat{a}^2\hat{\lambda}_i\} = 2\hat{a}\hat{R} + \hat{a}^2\hat{\lambda}_i,$$

then we have

$$\begin{aligned}\|\mathcal{K}\hat{q}_2 u\|_{H_0^2} &\le \left(\hat{a}\hat{R} - \hat{\lambda}_i + 4\sqrt{C_p}\hat{a}\hat{R} + 2C_p\hat{a}\hat{R} + C_p(2\hat{a}\hat{R} + \hat{a}^2\hat{\lambda}_i)\right)\|u\|\\ &= \left(\hat{a}\hat{R}(1 + 4\sqrt{C_p} + 4C_p) + \hat{\lambda}_i(-1 + C_p\hat{a}^2)\right)\|u\|.\end{aligned}$$

Also it hold that

$$\begin{aligned}\|\mathcal{K}\hat{q}_3 u\|_{H_0^2} \leq& \left\|\mathcal{K}\left(\hat{R}-\hat{\mu}_i-\hat{R}x^2\right)D^2u\right\|_{H_0^2}+\left\|\mathcal{K}\left(-\hat{R}(3\hat{a}^2-2)+\hat{a}(\hat{a}\hat{\mu}_i+2\hat{\lambda}_i)+3\hat{a}^2\hat{R}x^2\right)u\right\|_{H_0^2},\\ \leq&\left(\left\|\hat{R}-\hat{\mu}_i-\hat{R}x^2\right\|_\infty+2\sqrt{C_p}\left\|(\hat{R}-\hat{\mu}_i-\hat{R}x^2)'\right\|_\infty+C_p\left\|(\hat{R}-\hat{\mu}_i-\hat{R}x^2)''\right\|_\infty\right)\|u\|\\ &+C_p\left\|-\hat{R}(3\hat{a}^2-2)+\hat{a}(\hat{a}\hat{\mu}_i+2\hat{\lambda}_i)+3\hat{a}^2\hat{R}x^2\right\|_\infty\|u\|\\ =&\left(\left\|\hat{R}-\hat{\mu}_i-\hat{R}x^2\right\|_\infty+2\sqrt{C_p}\left\|-2\hat{R}x\right\|_\infty+C_p\left\|-2\hat{R}\right\|_\infty\right)\|u\|\\ &+C_p\left\|-3\hat{a}^2\hat{R}+2\hat{R}+\hat{a}(\hat{a}\hat{\mu}_i+2\hat{\lambda}_i)+3\hat{a}^2\hat{R}x^2\right\|_\infty\|u\|\\ =&\left(\left\|\hat{R}-\hat{\mu}_i-\hat{R}x^2\right\|_\infty+4\sqrt{C_p}\hat{R}+2C_p\hat{R}\right)\|u\|\\ &+C_p\left\|-3\hat{a}^2\hat{R}+2\hat{R}+\hat{a}(\hat{a}\hat{\mu}_i+2\hat{\lambda}_i)+3\hat{a}^2\hat{R}x^2\right\|_\infty\|u\|,\end{aligned}$$

and noting that

$$\Omega=[-1,1],\quad \hat{\lambda}_i\approx 1555,\quad \hat{a}\approx 1.020,\quad \hat{R}\approx 5772,\quad \hat{\mu}_i\approx 2211,$$

we find that

$$\left\|\hat{R}-\hat{\mu}_i-\hat{R}x^2\right\|_\infty=\max\{\hat{R}-\hat{\mu}_i,\ \hat{\mu}_i\}=\hat{R}-\hat{\mu}_i,$$

$$-3\hat{a}^2\hat{R}+2\hat{R}+\hat{a}(\hat{a}\hat{\mu}_i+2\hat{\lambda}_i)\approx -1000,\qquad 2\hat{R}+\hat{a}(\hat{a}\hat{\mu}_i+2\hat{\lambda}_i)\approx 17016,$$

and

$$\begin{aligned}\left\|-3\hat{a}^2\hat{R}+2\hat{R}+\hat{a}(\hat{a}\hat{\mu}_i+2\hat{\lambda}_i)+3\hat{a}^2\hat{R}x^2\right\|_\infty&=\max\{|-3\hat{a}^2\hat{R}+2\hat{R}+\hat{a}(\hat{a}\hat{\mu}_i+2\hat{\lambda}_i)|,\ 2\hat{R}+\hat{a}(\hat{a}\hat{\mu}_i+2\hat{\lambda}_i)\}\\ &=2\hat{R}+\hat{a}(\hat{a}\hat{\mu}_i+2\hat{\lambda}_i),\end{aligned}$$

then

$$\begin{aligned}\|\mathcal{K}\hat{q}_3u\|_{H_0^2}\leq&\left(\hat{R}-\hat{\mu}_i+4\sqrt{C_p}\hat{R}+2C_p\hat{R}+C_p(2\hat{R}+\hat{a}(\hat{a}\hat{\mu}_i+2\hat{\lambda}_i)\right)\|u\|\\ =&\left(\hat{R}(1+4\sqrt{C_p}+4C_p)+C_p\hat{a}(\hat{a}\hat{\mu}_i+2\hat{\lambda}_i)-\hat{\mu}_i\right)\|u\|.\end{aligned}$$

We also obtain

$$\begin{aligned}\|\mathcal{K}\hat{q}_4u\|_{H_0^2}&\leq 4\hat{a}\left\|\mathcal{K}D^2u\right\|_{H_0^2}+4\hat{a}^3\|\mathcal{K}u\|_{H_0^2}\\ &\leq 4\hat{a}\|u\|+4C_p\hat{a}^3\|u\|\\ &=4\hat{a}(1+C_p\hat{a}^2)\|u\|.\end{aligned}$$

Consequently, by setting

$$s_1:=\hat{a}^2(2+\hat{a}^2C_p), \tag{9}$$
$$s_2:=\hat{a}\hat{R}(1+4\sqrt{C_p}+4C_p)+\hat{\lambda}_i(-1+C_p\hat{a}^2), \tag{10}$$
$$s_3:=\hat{R}(1+4\sqrt{C_p}+4C_p)+C_p\hat{a}(\hat{a}\hat{\mu}_i+2\hat{\lambda}_i)-\hat{\mu}_i, \tag{11}$$
$$s_4:=4\hat{a}(1+C_p\hat{a}^2), \tag{12}$$

and by using

$$\|u\|\leq C(N)\|u\|_{H_0^2}\leq C(N)\|w_*\|_X,\qquad u\in\{u_r^\perp,u_i^\perp,v_r^\perp,v_i^\perp\},$$

we obtain

$$\|P_h\mathscr{A}^{-1}\mathscr{Q}w_*\|_X^2\leq C(N)^2(2(s_1+s_2)^2+2(s_1+s_2+s_3+s_4)^2)\|w_*\|_X^2,$$

then we can take

$$\nu_1=C(N)\sqrt{2}\sqrt{(s_1+s_2)^2+(s_1+s_2+s_3+s_4)^2}.$$

INTLAB code

“compute_nu1.m”

```
format long;
intvalinit('DisplayInfsup');

K = 200;
C = sqrt(c3(2*K+2));
Cp = 4/intval('pi')^2;
fprintf('      Constant C(N):%15.5e\n',sup(C))

[a,R,lambda,mu,ur,ui,vr,vi] = input_approximate_solution(K);

s1 = a^2*( 2 + a^2*Cp );
s2 = a*R*( 1 + 4*sqrt(Cp) + 4*Cp ) + lambda*( -1 + Cp*a^2 );
s3 = R*( 1 + 4*sqrt(Cp) + 4*Cp ) + Cp*a*(a*mu + 2*lambda) - mu;
s4 = 4*a*( 1 + Cp*a^2 );

nu1 = C*sqrt(intval('2'))*sqrt( (s1+s2)^2 + (s1+s2+s3+s4)^2 );
fprintf('                 nu1:%15.5e\n',sup(nu1))
```

```
>> compute_nu1
===> Default display of intervals by infimum/supremum (e.g. [ 3.14 , 3.15 ])
      Constant C(N):    6.14232e-06
                nu1:    5.76194e-01
>> nu1
intval nu1 =
[   0.57619422625840,   0.57619422625841]
```

2 Bound of ν_2

For all $w \in X$, if we can estimate ν_2 and ν_3 satisfying

$$\|qP_h w\|_Y \le \nu_2 \|P_h w\|_X, \qquad \|q(I-P_h)w\|_Y \le \nu_3 \|(I-P_h)w\|_X$$

separately, we have

$$\begin{aligned}\|qw\|_Y &\le \|qP_h w\|_Y + \|q(I-P_h)w\|_Y \\ &\le \nu_2\|P_h w\|_X + \nu_3\|(I-P_h)w\|_X.\end{aligned}$$

In this section we consider an upper bound ν_2 for $w = [u_r, u_i, v_r, v_i, a, R, \lambda_i, \mu_i]^T \in S_h^4 \times \mathbb{R}^4$.
Because of

$$qw = \begin{bmatrix} f'[0]w \\ a - \left(u_r, u_r^0\right)_{H_0^2} \\ R - \left(u_i, u_i^0\right)_{H_0^2} \\ \lambda_i - \left(v_r, v_r^0\right)_{H_0^2} \\ \mu_i - \left(v_i, v_i^0\right)_{H_0^2} \end{bmatrix},$$

the norm of qw in Y is

$$\|qw\|_Y^2 = \|f'[0]w\|^2_{(L^2(\Omega))^4} + (a - \left(u_r, u_r^0\right)_{H_0^2})^2 + (R - \left(u_i, u_i^0\right)_{H_0^2})^2 + (\lambda_i - \left(v_r, v_r^0\right)_{H_0^2})^2 + (\mu_i - \left(v_i, v_i^0\right)_{H_0^2})^2.$$

We set

$$
\begin{aligned}
u_r &= \sum_{n=1}^{N} u_{r,n}\phi_n, & \boldsymbol{u}_r &= [u_{r,n}] \in \mathbb{R}^N, & u_i &= \sum_{n=1}^{N} u_{i,n}\phi_n, & \boldsymbol{u}_i &= [u_{i,n}] \in \mathbb{R}^N,\\
v_r &= \sum_{n=1}^{N} v_{r,n}\phi_n, & \boldsymbol{v}_r &= [v_{r,n}] \in \mathbb{R}^N, & v_i &= \sum_{n=1}^{N} v_{i,n}\phi_n, & \boldsymbol{v}_i &= [v_{i,n}] \in \mathbb{R}^N,\\
u_r^0 &= \sum_{n=1}^{N} u_{r,n}^0\phi_n, & \boldsymbol{u}_r^0 &= [u_{r,n}^0] \in \mathbb{R}^N, & u_i^0 &= \sum_{n=1}^{N} u_{i,n}^0\phi_n, & \boldsymbol{u}_i^0 &= [u_{i,n}^0] \in \mathbb{R}^N,\\
v_r^0 &= \sum_{n=1}^{N} v_{r,n}^0\phi_n, & \boldsymbol{v}_r^0 &= [v_{r,n}^0] \in \mathbb{R}^N, & v_i^0 &= \sum_{n=1}^{N} v_{i,n}^0\phi_n, & \boldsymbol{v}_i^0 &= [v_{i,n}^0] \in \mathbb{R}^N,
\end{aligned}
$$

and

$$
\boldsymbol{w} := [\boldsymbol{u}_r^T, \boldsymbol{u}_i^T, \boldsymbol{v}_r^T, \boldsymbol{v}_i^T, a, R, \lambda_i, \mu_i]^T \in \mathbb{R}^{4N+4}.
$$

For example, for $v_i, v_i^0 \in S_h$, using the orthogonality of the inner-product $(\,\cdot,\cdot\,)_{H_0^2}$, we have

$$
\begin{aligned}
[\boldsymbol{v}_i^T, \mu_i]\begin{bmatrix} \boldsymbol{v}_i^0(\boldsymbol{v}_i^0)^T & -\boldsymbol{v}_i^0 \\ -(\boldsymbol{v}_i^0)^T & 1 \end{bmatrix}\begin{bmatrix} \boldsymbol{v}_i \\ \mu_i \end{bmatrix} &= [\boldsymbol{v}_i^T, \mu_i]\begin{bmatrix} \boldsymbol{v}_i^0(\boldsymbol{v}_i^0)^T\boldsymbol{v}_i - \boldsymbol{v}_i^0\mu_i \\ -(\boldsymbol{v}_i^0)^T\boldsymbol{v}_i + \mu_i \end{bmatrix} = \boldsymbol{v}_i^T\boldsymbol{v}_i^0(\boldsymbol{v}_i^0)^T\boldsymbol{v}_i - \boldsymbol{v}_i^T\boldsymbol{v}_i^0\mu_i - \mu_i(\boldsymbol{v}_i^0)^T\boldsymbol{v}_i + \mu_i^2\\
&= ((\boldsymbol{v}_i^0)^T\boldsymbol{v}_i)^2 - 2\mu_i(\boldsymbol{v}_i^0)^T\boldsymbol{v}_i + \mu_i^2 = (\mu_i - (\, v_i, v_i^0\,)_{H_0^2})^2.
\end{aligned}
$$

Then it can be checked that

$$
\begin{aligned}
(a - (\, u_r, u_r^0\,)_{H_0^2})^2 &= [\boldsymbol{u}_r^T, a]\begin{bmatrix} \boldsymbol{u}_r^0(\boldsymbol{u}_r^0)^T & -\boldsymbol{u}_r^0 \\ -(\boldsymbol{u}_r^0)^T & 1 \end{bmatrix}\begin{bmatrix} \boldsymbol{u}_r \\ a \end{bmatrix},\\
(R - (\, u_i, u_i^0\,)_{H_0^2})^2 &= [\boldsymbol{u}_i^T, R]\begin{bmatrix} \boldsymbol{u}_i^0(\boldsymbol{u}_i^0)^T & -\boldsymbol{u}_i^0 \\ -(\boldsymbol{u}_i^0)^T & 1 \end{bmatrix}\begin{bmatrix} \boldsymbol{u}_i \\ R \end{bmatrix},\\
(\lambda_i - (\, v_r, v_r^0\,)_{H_0^2})^2 &= [\boldsymbol{v}_r^T, \lambda_i]\begin{bmatrix} \boldsymbol{v}_r^0(\boldsymbol{v}_r^0)^T & -\boldsymbol{v}_r^0 \\ -(\boldsymbol{v}_r^0)^T & 1 \end{bmatrix}\begin{bmatrix} \boldsymbol{v}_r \\ \lambda_i \end{bmatrix},\\
(\mu_i - (\, v_i, v_i^0\,)_{H_0^2})^2 &= [\boldsymbol{v}_i^T, \mu_i]\begin{bmatrix} \boldsymbol{v}_i^0(\boldsymbol{v}_i^0)^T & -\boldsymbol{v}_i^0 \\ -(\boldsymbol{v}_i^0)^T & 1 \end{bmatrix}\begin{bmatrix} \boldsymbol{v}_i \\ \mu_i \end{bmatrix},
\end{aligned}
$$

and

$$
\begin{aligned}
&(a - (\, u_r, u_r^0\,)_{H_0^2})^2 + (R - (\, u_i, u_i^0\,)_{H_0^2})^2 + (\lambda_i - (\, v_r, v_r^0\,)_{H_0^2})^2 + (\mu_i - (\, v_i, v_i^0\,)_{H_0^2})^2\\
&= \boldsymbol{w}^T\begin{bmatrix}
\boldsymbol{u}_r^0(\boldsymbol{u}_r^0)^T & 0 & 0 & 0 & -\boldsymbol{u}_r^0 & 0 & 0 & 0\\
0 & \boldsymbol{u}_i^0(\boldsymbol{u}_i^0)^T & 0 & 0 & 0 & -\boldsymbol{u}_i^0 & 0 & 0\\
0 & 0 & \boldsymbol{v}_r^0(\boldsymbol{v}_r^0)^T & 0 & 0 & 0 & -\boldsymbol{v}_r^0 & 0\\
0 & 0 & 0 & \boldsymbol{v}_i^0(\boldsymbol{v}_i^0)^T & 0 & 0 & 0 & -\boldsymbol{v}_i^0\\
-(\boldsymbol{u}_r^0)^T & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & -(\boldsymbol{u}_i^0)^T & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & -(\boldsymbol{v}_r^0)^T & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & -(\boldsymbol{v}_i^0)^T & 0 & 0 & 0 & 1
\end{bmatrix}\boldsymbol{w}.
\end{aligned}
$$

Since

$$
f'[0]w = s_u\begin{bmatrix} q_1'^* & -q_2'^* & 0 & 0\\ q_2'^* & q_1'^* & 0 & 0\\ q_4'^* & -q_3'^* & q_1'^* & -q_2'^*\\ q_3'^* & q_4'^* & q_2'^* & q_1'^* \end{bmatrix}\begin{bmatrix} \hat{u}_r\\ \hat{u}_i\\ \hat{v}_r\\ \hat{v}_i \end{bmatrix} + \begin{bmatrix} \hat{q}_1 & -\hat{q}_2 & 0 & 0\\ \hat{q}_2 & \hat{q}_1 & 0 & 0\\ \hat{q}_4 & -\hat{q}_3 & \hat{q}_1 & -\hat{q}_2\\ \hat{q}_3 & \hat{q}_4 & \hat{q}_2 & \hat{q}_1 \end{bmatrix}\begin{bmatrix} u_r\\ u_i\\ v_r\\ v_i \end{bmatrix}
$$

and

$$
\begin{aligned}
q_1'^* &= (q_1^*)_a[0]a,\\
q_2'^* &= (q_2^*)_a[0]a + (q_2^*)_R[0]R + (q_2^*)_{\lambda_i}[0]\lambda_i,\\
q_3'^* &= (q_3^*)_a[0]a + (q_3^*)_R[0]R + (q_3^*)_{\lambda_i}[0]\lambda_i + (q_3^*)_{\mu_i}[0]\mu_i,\\
q_4'^* &= (q_4^*)_a[0]a,
\end{aligned}
$$

the term $\|f'[0]w\|^2_{(L^2(\Omega))^4}$ is

$$\begin{aligned}\|f'[0]w\|^2_{(L^2(\Omega))^4} = \|s_u(q_1'^*\hat{u}_r - q_2'^*\hat{u}_i) + \hat{q}_1 u_r - \hat{q}_2 u_i\|^2 \\ + \|s_u(q_2'^*\hat{u}_r + q_1'^*\hat{u}_i) + \hat{q}_2 u_r + \hat{q}_1 u_i\|^2 \\ + \|s_u(q_4'^*\hat{u}_r - q_3'^*\hat{u}_i + q_1'^*\hat{v}_r - q_2'^*\hat{v}_i) + \hat{q}_4 u_r - \hat{q}_3 u_i + \hat{q}_1 v_r - \hat{q}_2 v_i\|^2 \\ + \|s_u(q_3'^*\hat{u}_r + q_4'^*\hat{u}_i + q_2'^*\hat{v}_r + q_1'^*\hat{v}_i) + \hat{q}_3 u_r + \hat{q}_4 u_i + \hat{q}_2 v_r + \hat{q}_1 v_i\|^2.\end{aligned}$$

Then using the facts of

$$\begin{aligned}q_1'^*\hat{u}_r - q_2'^*\hat{u}_i &= (q_1^*)_a[0]a\hat{u}_r - \Big((q_2^*)_a[0]a + (q_2^*)_R[0]R + (q_2^*)_{\lambda_i}[0]\lambda_i\Big)\hat{u}_i \\ &= \Big((q_1^*)_a[0]\hat{u}_r - (q_2^*)_a[0]\hat{u}_i\Big)a + (-(q_2^*)_R[0]\hat{u}_i)R + (-(q_2^*)_{\lambda_i}[0]\hat{u}_i)\lambda_i,\end{aligned}$$

$$\begin{aligned}q_2'^*\hat{u}_r + q_1'^*\hat{u}_i &= \Big((q_2^*)_a[0]a + (q_2^*)_R[0]R + (q_2^*)_{\lambda_i}[0]\lambda_i\Big)\hat{u}_r + (q_1^*)_a[0]a\hat{u}_i \\ &= \Big((q_2^*)_a[0]\hat{u}_r + (q_1^*)_a[0]\hat{u}_i\Big)a + (q_2^*)_R[0]\hat{u}_r R + (q_2^*)_{\lambda_i}[0]\hat{u}_r\lambda_i,\end{aligned}$$

$$\begin{aligned}q_4'^*\hat{u}_r - q_3'^*\hat{u}_i + q_1'^*\hat{v}_r - q_2'^*\hat{v}_i &= (q_4^*)_a[0]a\hat{u}_r - \Big((q_3^*)_a[0]a + (q_3^*)_R[0]R + (q_3^*)_{\lambda_i}[0]\lambda_i + (q_3^*)_{\mu_i}[0]\mu_i\Big)\hat{u}_i \\ &\quad + (q_1^*)_a[0]a\hat{v}_r - \Big((q_2^*)_a[0]a + (q_2^*)_R[0]R + (q_2^*)_{\lambda_i}[0]\lambda_i\Big)\hat{v}_i \\ &= \Big((q_4^*)_a[0]\hat{u}_r - (q_3^*)_a[0]\hat{u}_i + (q_1^*)_a[0]\hat{v}_r - (q_2^*)_a[0]\hat{v}_i\Big)a \\ &\quad + \Big(-(q_3^*)_R[0]\hat{u}_i - (q_2^*)_R[0]\hat{v}_i\Big)R + \Big(-(q_3^*)_{\lambda_i}[0]\hat{u}_i - (q_2^*)_{\lambda_i}[0]\hat{v}_i\Big)\lambda_i \\ &\quad + (-(q_3^*)_{\mu_i}[0]\hat{u}_i)\mu_i,\end{aligned}$$

and

$$\begin{aligned}q_3'^*\hat{u}_r + q_4'^*\hat{u}_i + q_2'^*\hat{v}_r + q_1'^*\hat{v}_i &= \Big((q_3^*)_a[0]a + (q_3^*)_R[0]R + (q_3^*)_{\lambda_i}[0]\lambda_i + (q_3^*)_{\mu_i}[0]\mu_i\Big)\hat{u}_r + (q_4^*)_a[0]a\hat{u}_i \\ &\quad + \Big((q_2^*)_a[0]a + (q_2^*)_R[0]R + (q_2^*)_{\lambda_i}[0]\lambda_i\Big)\hat{v}_r + (q_1^*)_a[0]a\hat{v}_i \\ &= \Big((q_3^*)_a[0]\hat{u}_r + (q_4^*)_a[0]\hat{u}_i + (q_2^*)_a[0]\hat{v}_r + (q_1^*)_a[0]\hat{v}_i\Big)a \\ &\quad + \Big((q_3^*)_R[0]\hat{u}_r + (q_2^*)_R[0]\hat{v}_r\Big)R + \Big((q_3^*)_{\lambda_i}[0]\hat{u}_r + (q_2^*)_{\lambda_i}[0]\hat{v}_r\Big)\lambda_i \\ &\quad + (q_3^*)_{\mu_i}[0]\hat{u}_r\mu_i,\end{aligned}$$

by defineing $z_k \in L^2(\Omega)$ $(k = 1, \dots, 14)$ by

$$
\begin{aligned}
z_1 &:= s_u\Big((q_1^*)_a[0]\hat{u}_r - (q_2^*)_a[0]\hat{u}_i\Big),\\
z_2 &:= s_u\Big(-(q_2^*)_R[0]\hat{u}_i\Big),\\
z_3 &:= s_u\Big(-(q_2^*)_{\lambda_i}[0]\hat{u}_i\Big),\\
z_4 &:= s_u\Big((q_2^*)_a[0]\hat{u}_r + (q_1^*)_a[0]\hat{u}_i\Big),\\
z_5 &:= s_u\Big((q_2^*)_R[0]\hat{u}_r\Big),\\
z_6 &:= s_u\Big((q_2^*)_{\lambda_i}[0]\hat{u}_r\Big),\\
z_7 &:= s_u\Big((q_4^*)_a[0]\hat{u}_r - (q_3^*)_a[0]\hat{u}_i + (q_1^*)_a[0]\hat{v}_r - (q_2^*)_a[0]\hat{v}_i\Big),\\
z_8 &:= s_u\Big(-(q_3^*)_R[0]\hat{u}_i - (q_2^*)_R[0]\hat{v}_i\Big),\\
z_9 &:= s_u\Big(-(q_3^*)_{\lambda_i}[0]\hat{u}_i - (q_2^*)_{\lambda_i}[0]\hat{v}_i\Big),\\
z_{10} &:= s_u\Big(-(q_3^*)_{\mu_i}[0]\hat{u}_i\Big),\\
z_{11} &:= s_u\Big((q_3^*)_a[0]\hat{u}_r + (q_4^*)_a[0]\hat{u}_i + (q_2^*)_a[0]\hat{v}_r + (q_1^*)_a[0]\hat{v}_i\Big),\\
z_{12} &:= s_u\Big((q_3^*)_R[0]\hat{u}_r + (q_2^*)_R[0]\hat{v}_r\Big),\\
z_{13} &:= s_u\Big((q_3^*)_{\lambda_i}[0]\hat{u}_r + (q_2^*)_{\lambda_i}[0]\hat{v}_r\Big),\\
z_{14} &:= s_u\Big((q_3^*)_{\mu_i}[0]\hat{u}_r\Big),
\end{aligned}
$$

we can rewrite

$$
\begin{aligned}
s_u(q_1'^*\hat{u}_r - q_2'^*\hat{u}_i) &= z_1 a + z_2 R + z_3\lambda_i,\\
s_u(q_2'^*\hat{u}_r + q_1'^*\hat{u}_i) &= z_4 a + z_5 R + z_6\lambda_i,\\
s_u(q_4'^*\hat{u}_r - q_3'^*\hat{u}_i + q_1'^*\hat{v}_r - q_2'^*\hat{v}_i) &= z_7 a + z_8 R + z_9\lambda_i + z_{10}\mu_i,\\
s_u(q_3'^*\hat{u}_r + q_4'^*\hat{u}_i + q_2'^*\hat{v}_r + q_1'^*\hat{v}_i) &= z_{11} a + z_{12} R + z_{13}\lambda_i + z_{14}\mu_i,
\end{aligned}
$$

and the term $\|f'[0]w\|^2_{(L^2(\Omega))^4}$ is

$$
\begin{aligned}
\|f'[0]w\|^2_{(L^2(\Omega))^4} = {} & \|z_1 a + z_2 R + z_3\lambda_i + \hat{q}_1 u_r - \hat{q}_2 u_i\|^2\\
&+ \|z_4 a + z_5 R + z_6\lambda_i + \hat{q}_2 u_r + \hat{q}_1 u_i\|^2\\
&+ \|z_7 a + z_8 R + z_9\lambda_i + z_{10}\mu_i + \hat{q}_4 u_r - \hat{q}_3 u_i + \hat{q}_1 v_r - \hat{q}_2 v_i\|^2\\
&+ \|z_{11} a + z_{12} R + z_{13}\lambda_i + z_{14}\mu_i + \hat{q}_3 u_r + \hat{q}_4 u_i + \hat{q}_2 v_r + \hat{q}_1 v_i\|^2.
\end{aligned}
$$

Now, it holds that

$$
\begin{aligned}
z_1 a + z_2 R + z_3\lambda_i + \hat{q}_1 u_r - \hat{q}_2 u_i &= \sum_{n=1}^N u_{r,n}\hat{q}_1\phi_n - \sum_{n=1}^N u_{i,n}\,\hat{q}_2\,\phi_n + z_1 a + z_2 R + z_3\lambda_i,\\
z_4 a + z_5 R + z_6\lambda_i + \hat{q}_2 u_r + \hat{q}_1 u_i &= \sum_{n=1}^N u_{r,n}\hat{q}_2\phi_n + \sum_{n=1}^N u_{i,n}\,\hat{q}_1\,\phi_n + z_4 a + z_5 R + z_6\lambda_i,
\end{aligned}
$$

$$
\begin{aligned}
&z_7 a + z_8 R + z_9\lambda_i + z_{10}\mu_i + \hat{q}_4 u_r - \hat{q}_3 u_i + \hat{q}_1 v_r - \hat{q}_2 v_i\\
&= \sum_{n=1}^N u_{r,n}\,\hat{q}_4\,\phi_n - \sum_{n=1}^N u_{i,n}\,\hat{q}_3\,\phi_n + \sum_{n=1}^N v_{r,n}\,\hat{q}_1\,\phi_n - \sum_{n=1}^N v_{i,n}\,\hat{q}_2\,\phi_n + z_7 a + z_8 R + z_9\lambda_i + z_{10}\mu_i,
\end{aligned}
$$

$$
\begin{aligned}
&z_{11}a + z_{12}R + z_{13}\lambda_i + z_{14}\mu_i + \hat{q}_3 u_r + \hat{q}_4 u_i + \hat{q}_2 v_r + \hat{q}_1 v_i \\
&= \sum_{n=1}^{N} u_{r,n}\,\hat{q}_3\,\phi_n + \sum_{n=1}^{N} u_{i,n}\,\hat{q}_4\,\phi_n + \sum_{n=1}^{N} v_{r,n}\,\hat{q}_2\,\phi_n + \sum_{n=1}^{N} v_{i,n}\,\hat{q}_1\,\phi_n + z_{11}a + z_{12}R + z_{13}\lambda_i + z_{14}\mu_i.
\end{aligned}
$$

Then by defining

$$
\begin{aligned}
&p_n^1 := \hat{q}_1\,\phi_n, \quad p_n^2 := \hat{q}_2\,\phi_n, \quad p_n^3 := \hat{q}_3\,\phi_n, \quad p_n^4 := \hat{q}_4\,\phi_n, \qquad 1 \le n \le N, \\
&\boldsymbol{z}_k^l := [(\, z_k, \hat{q}_l \phi_m \,)_{L^2}] = [(\, z_k, p_m^l \,)_{L^2}] \in \mathbb{R}^N, \qquad 1 \le k \le 14, \quad 1 \le l \le 4,
\end{aligned}
$$

and

$$
[C_{i,j}]_{m,n} := \left(p_n^i, p_m^j \right)_{L^2} \in \mathbb{R}^{N \times N}, \qquad 1 \le i, j \le 4,
$$

it is true that

$$
\begin{aligned}
&\|z_1 a + z_2 R + z_3 \lambda_i + \hat{q}_1 u_r - \hat{q}_2 u_i\|^2 \\
&= \boldsymbol{w}^T \begin{bmatrix}
C_{1,1} & -C_{1,2}^T & 0 & 0 & \boldsymbol{z}_1^1 & \boldsymbol{z}_2^1 & \boldsymbol{z}_3^1 & 0 \\
 & C_{2,2} & 0 & 0 & -\boldsymbol{z}_1^2 & -\boldsymbol{z}_2^2 & -\boldsymbol{z}_3^2 & 0 \\
 & & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & 0 & 0 & 0 & 0 & 0 \\
 & & & & \|z_1\|^2 & (\, z_1, z_2 \,)_{L^2} & (\, z_1, z_3 \,)_{L^2} & 0 \\
 & & & & & \|z_2\|^2 & (\, z_2, z_3 \,)_{L^2} & 0 \\
 & & (sym.) & & & & \|z_3\|^2 & 0 \\
 & & & & & & & 0
\end{bmatrix} \boldsymbol{w},
\end{aligned}
$$

$$
\begin{aligned}
&\|z_4 a + z_5 R + z_6 \lambda_i + \hat{q}_2 u_r + \hat{q}_1 u_i\|^2 \\
&= \boldsymbol{w}^T \begin{bmatrix}
C_{2,2} & C_{1,2} & 0 & 0 & \boldsymbol{z}_4^2 & \boldsymbol{z}_5^2 & \boldsymbol{z}_6^2 & 0 \\
 & C_{1,1} & 0 & 0 & \boldsymbol{z}_4^1 & \boldsymbol{z}_5^1 & \boldsymbol{z}_6^1 & 0 \\
 & & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & 0 & 0 & 0 & 0 & 0 \\
 & & & & \|z_4\|^2 & (\, z_4, z_5 \,)_{L^2} & (\, z_4, z_6 \,)_{L^2} & 0 \\
 & & & & & \|z_5\|^2 & (\, z_5, z_6 \,)_{L^2} & 0 \\
 & & (sym.) & & & & \|z_6\|^2 & 0 \\
 & & & & & & & 0
\end{bmatrix} \boldsymbol{w},
\end{aligned}
$$

$$
\begin{aligned}
&\|z_7 a + z_8 R + z_9 \lambda_i + z_{10}\mu_i + \hat{q}_4 u_r - \hat{q}_3 u_i + \hat{q}_1 v_r - \hat{q}_2 v_i\|^2 \\
&= \boldsymbol{w}^T \begin{bmatrix}
C_{4,4} & -C_{3,4} & C_{1,4} & -C_{2,4} & \boldsymbol{z}_7^4 & \boldsymbol{z}_8^4 & \boldsymbol{z}_9^4 & \boldsymbol{z}_{10}^4 \\
 & C_{3,3} & -C_{1,3} & C_{2,3} & -\boldsymbol{z}_7^3 & -\boldsymbol{z}_8^3 & -\boldsymbol{z}_9^3 & -\boldsymbol{z}_{10}^3 \\
 & & C_{1,1} & -C_{1,2}^T & \boldsymbol{z}_7^1 & \boldsymbol{z}_8^1 & \boldsymbol{z}_9^1 & \boldsymbol{z}_{10}^1 \\
 & & & C_{2,2} & -\boldsymbol{z}_7^2 & -\boldsymbol{z}_8^2 & -\boldsymbol{z}_9^2 & -\boldsymbol{z}_{10}^2 \\
 & & & & \|z_7\|^2 & (\, z_7, z_8 \,)_{L^2} & (\, z_7, z_9 \,)_{L^2} & (\, z_7, z_{10} \,)_{L^2} \\
 & & & & & \|z_8\|^2 & (\, z_8, z_9 \,)_{L^2} & (\, z_8, z_{10} \,)_{L^2} \\
 & & \text{sym.} & & & & \|z_9\|^2 & (\, z_9, z_{10} \,)_{L^2} \\
 & & & & & & & \|z_{10}\|^2
\end{bmatrix} \boldsymbol{w},
\end{aligned}
$$

$$
\begin{aligned}
&\|z_{11} a + z_{12} R + z_{13} \lambda_i + z_{14}\mu_i + \hat{q}_3 u_r + \hat{q}_4 u_i + \hat{q}_2 v_r + \hat{q}_1 v_i\|^2 \\
&= \boldsymbol{w}^T \begin{bmatrix}
C_{3,3} & C_{3,4}^T & C_{2,3} & C_{1,3} & \boldsymbol{z}_{11}^3 & \boldsymbol{z}_{12}^3 & \boldsymbol{z}_{13}^3 & \boldsymbol{z}_{14}^3 \\
 & C_{4,4} & C_{2,4} & C_{1,4} & \boldsymbol{z}_{11}^4 & \boldsymbol{z}_{12}^4 & \boldsymbol{z}_{13}^4 & \boldsymbol{z}_{14}^4 \\
 & & C_{2,2} & C_{1,2} & \boldsymbol{z}_{11}^2 & \boldsymbol{z}_{12}^2 & \boldsymbol{z}_{13}^2 & \boldsymbol{z}_{14}^2 \\
 & & & C_{1,1} & \boldsymbol{z}_{11}^1 & \boldsymbol{z}_{12}^1 & \boldsymbol{z}_{13}^1 & \boldsymbol{z}_{14}^1 \\
 & & & & \|z_{11}\|^2 & (\, z_{11}, z_{12} \,)_{L^2} & (\, z_{11}, z_{13} \,)_{L^2} & (\, z_{11}, z_{14} \,)_{L^2} \\
 & & & & & \|z_{12}\|^2 & (\, z_{12}, z_{13} \,)_{L^2} & (\, z_{12}, z_{14} \,)_{L^2} \\
 & & \text{sym.} & & & & \|z_{13}\|^2 & (\, z_{13}, z_{14} \,)_{L^2} \\
 & & & & & & & \|z_{14}\|^2
\end{bmatrix} \boldsymbol{w}.
\end{aligned}
$$

Hence we have

$$
\|qw\|_Y^2 = \boldsymbol{w}^T E \boldsymbol{w},
$$

where

$$E = \begin{bmatrix} E_{1,1} & E_{1,2} & E_{1,3} & E_{1,4} & E_{1,5} & E_{1,6} & E_{1,7} & E_{1,8} \\ & E_{2,2} & E_{2,3} & E_{2,4} & E_{2,5} & E_{2,6} & E_{2,7} & E_{2,8} \\ & & E_{3,3} & E_{3,4} & E_{3,5} & E_{3,6} & E_{3,7} & E_{3,8} \\ & & & E_{4,4} & E_{4,5} & E_{4,6} & E_{4,7} & E_{4,8} \\ & & & & E_{5,5} & E_{5,6} & E_{5,7} & E_{5,8} \\ & & & & & E_{6,6} & E_{6,7} & E_{6,8} \\ & \text{sym.} & & & & & E_{7,7} & E_{7,8} \\ & & & & & & & E_{8,8} \end{bmatrix}, \tag{13}$$

and

$$\begin{aligned}
E_{1,1} &= C_{1,1} + C_{2,2} + C_{3,3} + C_{4,4} + \boldsymbol{u}_r^0(\boldsymbol{u}_r^0)^T, \\
E_{1,2} &= -C_{1,2}^T + C_{1,2} - C_{3,4} + C_{3,4}^T, \\
E_{1,3} &= C_{1,4} + C_{2,3}, \\
E_{1,4} &= -C_{2,4} + C_{1,3}, \\
E_{2,2} &= C_{1,1} + C_{2,2} + C_{3,3} + C_{4,4} + \boldsymbol{u}_i^0(\boldsymbol{u}_i^0)^T, \\
E_{2,3} &= -C_{1,3} + C_{2,4}, \\
E_{2,4} &= C_{1,4} + C_{2,3}, \\
E_{3,3} &= C_{1,1} + C_{2,2} + \boldsymbol{v}_r^0(\boldsymbol{v}_r^0)^T, \\
E_{3,4} &= -C_{1,2}^T + C_{1,2}, \\
E_{4,4} &= C_{1,1} + C_{2,2} + \boldsymbol{v}_i^0(\boldsymbol{v}_i^0)^T, \\
E_{1,5} &= \boldsymbol{z}_1^1 + \boldsymbol{z}_4^2 + \boldsymbol{z}_7^4 + \boldsymbol{z}_{11}^3 - \boldsymbol{u}_r^0, \\
E_{1,6} &= \boldsymbol{z}_2^1 + \boldsymbol{z}_5^2 + \boldsymbol{z}_8^4 + \boldsymbol{z}_{12}^3, \\
E_{1,7} &= \boldsymbol{z}_3^1 + \boldsymbol{z}_6^2 + \boldsymbol{z}_9^4 + \boldsymbol{z}_{13}^3, \\
E_{1,8} &= \boldsymbol{z}_{10}^4 + \boldsymbol{z}_{14}^3, \\
E_{2,5} &= -\boldsymbol{z}_1^2 + \boldsymbol{z}_4^1 - \boldsymbol{z}_7^3 + \boldsymbol{z}_{11}^4, \\
E_{2,6} &= -\boldsymbol{z}_2^2 + \boldsymbol{z}_5^1 - \boldsymbol{z}_8^3 + \boldsymbol{z}_{12}^4 - \boldsymbol{u}_i^0, \\
E_{2,7} &= -\boldsymbol{z}_3^2 + \boldsymbol{z}_6^1 - \boldsymbol{z}_9^3 + \boldsymbol{z}_{13}^4, \\
E_{2,8} &= -\boldsymbol{z}_{10}^3 + \boldsymbol{z}_{14}^4, \\
E_{3,5} &= \boldsymbol{z}_7^1 + \boldsymbol{z}_{11}^2, \\
E_{3,6} &= \boldsymbol{z}_8^1 + \boldsymbol{z}_{12}^2, \\
E_{3,7} &= \boldsymbol{z}_9^1 + \boldsymbol{z}_{13}^2 - \boldsymbol{v}_r^0, \\
E_{3,8} &= \boldsymbol{z}_{10}^1 + \boldsymbol{z}_{14}^2, \\
E_{4,5} &= -\boldsymbol{z}_7^2 + \boldsymbol{z}_{11}^1, \\
E_{4,6} &= -\boldsymbol{z}_8^2 + \boldsymbol{z}_{12}^1, \\
E_{4,7} &= -\boldsymbol{z}_9^2 + \boldsymbol{z}_{13}^1, \\
E_{4,8} &= -\boldsymbol{z}_{10}^2 + \boldsymbol{z}_{14}^1 - \boldsymbol{v}_i^0, \\
E_{5,5} &= \|z_1\|^2 + \|z_4\|^2 + \|z_7\|^2 + \|z_{11}\|^2 + 1, \\
E_{5,6} &= (\, z_1, z_2 \,)_{L^2} + (\, z_4, z_5 \,)_{L^2} + (\, z_7, z_8 \,)_{L^2} + (\, z_{11}, z_{12} \,)_{L^2}, \\
E_{5,7} &= (\, z_1, z_3 \,)_{L^2} + (\, z_4, z_6 \,)_{L^2} + (\, z_7, z_9 \,)_{L^2} + (\, z_{11}, z_{13} \,)_{L^2}, \\
E_{5,8} &= (\, z_7, z_{10} \,)_{L^2} + (\, z_{11}, z_{14} \,)_{L^2}, \\
E_{6,6} &= \|z_2\|^2 + \|z_5\|^2 + \|z_8\|^2 + \|z_{12}\|^2 + 1, \\
E_{6,7} &= (\, z_2, z_3 \,)_{L^2} + (\, z_5, z_6 \,)_{L^2} + (\, z_8, z_9 \,)_{L^2} + (\, z_{12}, z_{13} \,)_{L^2}, \\
E_{6,8} &= (\, z_8, z_{10} \,)_{L^2} + (\, z_{12}, z_{14} \,)_{L^2}, \\
E_{7,7} &= \|z_3\|^2 + \|z_6\|^2 + \|z_9\|^2 + \|z_{13}\|^2 + 1, \\
E_{7,8} &= (\, z_9, z_{10} \,)_{L^2} + (\, z_{13}, z_{14} \,)_{L^2}, \\
E_{8,8} &= \|z_{10}\|^2 + \|z_{14}\|^2 + 1.
\end{aligned}$$

Consequently, if $w \neq 0$, since $\|w\|_X^2 = \boldsymbol{w}^T\boldsymbol{w}$ and E is symmetric and positive semi-definite, it holds that

$$\|qw\|_Y^2 = \boldsymbol{w}^T E \boldsymbol{w} = \frac{\boldsymbol{w}^T E \boldsymbol{w}}{\boldsymbol{w}^T \boldsymbol{w}} \boldsymbol{w}^T \boldsymbol{w} = \frac{\boldsymbol{w}^T E \boldsymbol{w}}{\boldsymbol{w}^T \boldsymbol{w}} \|w\|_X^2 = \|E\|_2 \cdot \|w\|_X^2,$$

then we can take

$$\nu_2 = \sqrt{\|E\|_2}. \tag{14}$$

2.1 Computation of C_{ij}

Because of

$$p_n^1 = 2\hat{a}^2 D^2\phi_n - \hat{a}^4\phi_n,$$

$$p_n^2 = (\hat{a}\hat{R} - \hat{\lambda}_i)D^2\phi_n - \hat{a}\hat{R}x^2 D^2\phi_n + \hat{a}(-\hat{R}(\hat{a}^2-2) + \hat{a}\hat{\lambda}_i)\phi_n + \hat{a}^3\hat{R}x^2\phi_n,$$

$$p_n^3 = (\hat{R} - \hat{\mu}_i)D^2\phi_m - \hat{R}x^2 D^2\phi_n + (-\hat{R}(3\hat{a}^2-2) + \hat{a}(\hat{a}\hat{\mu}_i + 2\hat{\lambda}_i))\phi_n + 3\hat{a}^2\hat{R}x^2\phi_n,$$

$$p_n^4 = 4\hat{a}D^2\phi_n - 4\hat{a}^3\phi_n,$$

p_n^k ($k = 1,\dots,4$) can be written as

$$p_n^k = k_1^n D^2\phi_n + k_2^n x^2 D^2\phi_n + k_3^n\phi_n + k_4^n x^2\phi_n.$$

Therefore, using

$$(\phi_n'', \phi_m)_{L^2} = (\phi_n, \phi_m'')_{L^2}, \qquad (x^2\phi_n, \phi_m)_{L^2} = (\phi_n, x^2\phi_m)_{L^2},$$

we have

$$C_{i,j} = (p_n^i, p_m^j)_{L^2} = (k_1^n\phi_n'' + k_2^n x^2\phi_n'' + k_3^n\phi_n + k_4^n x^2\phi_n, k_1^m\phi_m'' + k_2^m x^2\phi_m'' + k_3^m\phi_m + k_4^m x^2\phi_m)_{L^2}$$

$$= \begin{bmatrix} k_1^m \\ k_2^m \\ k_3^m \\ k_4^m \end{bmatrix}^T \begin{bmatrix} (\phi_n'', \phi_m'')_{L^2} & (\phi_n'', x^2\phi_m'')_{L^2} & (\phi_n'', \phi_m)_{L^2} & (\phi_n'', x^2\phi_m)_{L^2} \\ (x^2\phi_n'', \phi_m'')_{L^2} & (x^2\phi_n'', x^2\phi_m'')_{L^2} & (x^2\phi_n'', \phi_m)_{L^2} & (x^2\phi_n'', x^2\phi_m)_{L^2} \\ (\phi_n, \phi_m'')_{L^2} & (\phi_n, x^2\phi_m'')_{L^2} & (\phi_n, \phi_m)_{L^2} & (\phi_n, x^2\phi_m)_{L^2} \\ (x^2\phi_n, \phi_m'')_{L^2} & (x^2\phi_n, x^2\phi_m'')_{L^2} & (x^2\phi_n, \phi_m)_{L^2} & (x^2\phi_n, x^2\phi_m)_{L^2} \end{bmatrix} \begin{bmatrix} k_1^n \\ k_2^n \\ k_3^n \\ k_4^n \end{bmatrix}$$

$$= \begin{bmatrix} k_1^m \\ k_2^m \\ k_3^m \\ k_4^m \end{bmatrix}^T \begin{bmatrix} I & [B_{10}]_{mn} & [B_2]_{mn} & [B_3]_{mn} \\ [B_{10}]_{mn} & [B_{11}]_{mn} & [B_3]_{mn} & [B_{12}]_{mn} \\ [B_2]_{mn} & [B_3]_{mn}^T & [B_1]_{mn} & [B_4]_{mn}^T \\ [B_3]_{mn}^T & [B_{12}]_{mn}^T & [B_4]_{mn} & [B_7]_{mn} \end{bmatrix} \begin{bmatrix} k_1^n \\ k_2^n \\ k_3^n \\ k_4^n \end{bmatrix},$$

where

$$\begin{aligned}
[B_1]_{mn} &= (\phi_n, \phi_m)_{L^2}, \\
[B_2]_{mn} &= (\phi_n'', \phi_m)_{L^2}, \\
[B_3]_{mn} &= (x^2\phi_n'', \phi_m)_{L^2}, \\
[B_4]_{mn} &= (x^2\phi_n, \phi_m)_{L^2}, \\
[B_7]_{mn} &:= (x^2\phi_n, x^2\phi_m)_{L^2}, \\
[B_{10}]_{mn} &:= (\phi_n'', x^2\phi_m'')_{L^2}, \\
[B_{11}]_{mn} &:= (x^2\phi_n'', x^2\phi_m'')_{L^2}, \\
[B_{12}]_{mn} &:= (x^2\phi_n'', x^2\phi_m)_{L^2}
\end{aligned}$$

introduced in the proof of Lemma 4.1.

2.2 z_k and its inner-products

By setting

$$
\begin{aligned}
y_n^1 &:= s_u(q_1^*)_a[0]\phi_n = s_u(4\hat{a}D^2\phi_n - 4\hat{a}^3\phi_n),\\
y_n^2 &:= s_u(q_2^*)_a[0]\phi_n = s_u(\hat{R}D^2\phi_n - \hat{R}x^2D^2\phi_n + (-\hat{R}(3\hat{a}^2-2)+2\hat{a}\hat{\lambda}_i)\phi_n + 3\hat{a}^2\hat{R}x^2\phi_n),\\
y_n^3 &:= s_u(q_2^*)_R[0]\phi_n = s_u/s_r(\hat{a}D^2\phi_n - \hat{a}x^2D^2\phi_n + \hat{a}(-\hat{a}^2+2)\phi_n + \hat{a}^3x^2\phi_n),\\
y_n^4 &:= s_u(q_2^*)_{\lambda_i}[0]\phi_n = s_u/s_l(-D^2\phi_n + \hat{a}^2\phi_n),\\
y_n^5 &:= s_u(q_4^*)_a[0]\phi_n = s_u(4D^2\phi_n - 12\hat{a}^2\phi_n),\\
y_n^6 &:= s_u(q_3^*)_a[0]\phi_n = s_u(2(\hat{a}(-3\hat{R}+\hat{\mu}_i)+\hat{\lambda}_i)\phi_n + 6\hat{a}\hat{R}x^2\phi_n),\\
y_n^7 &:= s_u(q_3^*)_R[0]\phi_n = s_u/s_r(D^2\phi_n - x^2D^2\phi_n + (-3\hat{a}^2+2)\phi_n + 3\hat{a}^2x^2\phi_n),\\
y_n^8 &:= s_u(q_3^*)_{\lambda_i}[0]\phi_n = s_u/s_l(2\hat{a}\phi_n),\\
y_n^9 &:= s_u(q_3^*)_{\mu_i}[0]\phi_n = s_u/s_m(-D^2\phi_n + \hat{a}^2\phi_n),
\end{aligned}
$$

z_k $(k = 1, \cdots, 14)$ can be expressed

$$
z_k = \mathrm{sgn}(h_1)\sum_{n=1}^{N} u_{r,n}\, y_n^{h_1} + \mathrm{sgn}(h_2)\sum_{n=1}^{N} u_{i,n}\, y_n^{h_2} + \mathrm{sgn}(h_3)\sum_{n=1}^{N} v_{r,n}\, y_n^{h_3} + \mathrm{sgn}(h_4)\sum_{n=1}^{N} v_{i,n}\, y_n^{h_4}, \tag{15}
$$

where

$$
h_1, h_2, h_3, h_4 \in \{1, \cdots, 9\}
$$

and the value of "sgn" is in $\{-1, 0, 1\}$ (see a table in the program).
y_n^i $(l = 1, \ldots, 9)$ can be written as

$$
y_n^l = k_1^n D^2\phi_n + k_2^n x^2 D^2\phi_n + k_3^n\phi_n + k_4^n x^2\phi_n
$$

such that

	k_1^n	k_2^n	k_3^n	k_4^n
y_n^1	$4\hat{a}s_u$	0	$-4\hat{a}^3s_u$	0
y_n^2	$\hat{R}s_u$	$-\hat{R}s_u$	$(-\hat{R}(3\hat{a}^2-2)+2\hat{a}\hat{\lambda}_i)s_u$	$3\hat{a}^2\hat{R}s_u$
y_n^3	$\hat{a}s_u/s_r$	$-\hat{a}s_u/s_r$	$\hat{a}(-\hat{a}^2+2)s_u/s_r$	$\hat{a}^3s_u/s_r$
y_n^4	$-s_u/s_l$	0	$\hat{a}^2s_u/s_l$	0
y_n^5	$4s_u$	0	$-12\hat{a}^2s_u$	0
y_n^6	0	0	$2(\hat{a}(-3\hat{R}+\hat{\mu}_i)+\hat{\lambda}_i)s_u$	$6\hat{a}\hat{R}s_u$
y_n^7	s_u/s_r	$-s_u/s_r$	$(-3\hat{a}^2+2)s_u/s_r$	$3\hat{a}^2s_u/s_r$
y_n^8	0	0	$2\hat{a}s_u/s_l$	0
y_n^9	$-s_u/s_m$	0	$\hat{a}^2s_u/s_m$	0

and matrices

$$
[D_{i,j}]_{m,n} := \left(y_n^i, y_m^j \right)_{L^2}, \qquad 1 \le i, j \le 9
$$

can be determined by

$$
\left(y_n^i, y_m^j \right)_{L^2} = \begin{bmatrix} k_1^m \\ k_2^m \\ k_3^m \\ k_4^m \end{bmatrix}^T \begin{bmatrix} I & [B_{10}]_{mn} & [B_2]_{mn} & [B_3]_{mn} \\ [B_{10}]_{mn} & [B_{11}]_{mn} & [B_3]_{mn} & [B_{12}]_{mn} \\ [B_2]_{mn} & [B_3]_{mn}^T & [B_1]_{mn} & [B_4]_{mn}^T \\ [B_3]_{mn}^T & [B_{12}]_{mn}^T & [B_4]_{mn} & [B_7]_{mn} \end{bmatrix} \begin{bmatrix} k_1^n \\ k_2^n \\ k_3^n \\ k_4^n \end{bmatrix}.
$$

Therefore inner-products

$$
\left(z_k, z_l \right)_{L^2}
$$

can be computed by using vectors $\boldsymbol{u}_r, \boldsymbol{u}_i, \boldsymbol{v}_r, \boldsymbol{v}_i$ with sgn function.

2.3 Vectors z_k^l

For vectors

$$\boldsymbol{z}_k^l = [(\, z_k, \hat{q}_l\phi_m\,)_{L^2}] = [\big(\, z_k, p_m^l\,\big)_{L^2}] \in \mathbb{R}^N, \qquad 1 \le k \le 14, \quad 1 \le l \le 4, \quad 1 \le m \le N,$$

from (15), we have

$$\boldsymbol{z}_k^l = \text{sgn}(h_1)F_{h_1,l}\,\boldsymbol{u}_r + \text{sgn}(h_2)F_{h_2,l}\,\boldsymbol{u}_i + \text{sgn}(h_3)F_{h_3,l}\,\boldsymbol{v}_r + \text{sgn}(h_4)F_{h_4,l}\,\boldsymbol{v}_i,$$

where

$$[F_{s,l}]_{m,n} = \big(\, p_n^l, y_m^s\,\big)_{L^2}, \qquad s \in \{h_1, h_2, h_3, h_4\}.$$

Because of

$$p_n^1 = 2\hat{a}^2 D^2\phi_n - \hat{a}^4\phi_n,$$

$$p_n^2 = (\hat{a}\hat{R} - \hat{\lambda}_i)D^2\phi_n - \hat{a}\hat{R}x^2D^2\phi_n + \hat{a}(-\hat{R}(\hat{a}^2-2) + \hat{a}\hat{\lambda}_i)\phi_n + \hat{a}^3\hat{R}x^2\phi_n,$$

$$p_n^3 = (\hat{R} - \hat{\mu}_i)D^2\phi_m - \hat{R}x^2D^2\phi_n + (-\hat{R}(3\hat{a}^2-2) + \hat{a}(\hat{a}\hat{\mu}_i + 2\hat{\lambda}_i))\phi_n + 3\hat{a}^2\hat{R}x^2\phi_n,$$

$$p_n^4 = 4\hat{a}D^2\phi_n - 4\hat{a}^3\phi_n,$$

p_n^k $(k = 1, 2, 3, 4)$ can be written as

$$p_n^k = \hat{k}_1^n D^2\phi_n + \hat{k}_2^n x^2 D^2\phi_n + \hat{k}_3^n\phi_n + \hat{k}_4^n x^2\phi_n$$

such that

	$\hat{k}_1^n$	$\hat{k}_2^n$	$\hat{k}_3^n$	$\hat{k}_4^n$
p_n^1	$2\hat{a}^2$	0	$-\hat{a}^4$	0
p_n^2	$\hat{a}\hat{R} - \hat{\lambda}_i$	$-\hat{a}\hat{R}$	$\hat{a}(-\hat{R}(a^2-2) + \hat{a}\hat{\lambda}_i)$	$\hat{a}^3\hat{R}$
p_n^3	$\hat{R} - \hat{\mu}_i$	$-\hat{R}$	$-\hat{R}(3\hat{a}^2-2) + \hat{a}(\hat{a}\hat{\mu}_i + 2\hat{\lambda}_i)$	$3\hat{a}^2\hat{R}$
p_n^4	$4\hat{a}$	0	$-4\hat{a}^3$	0

.

Therefore, we have

$$\big(\, p_n^i, y_m^j\,\big)_{L^2} = \Big(\, \hat{k}_1^n\phi_n'' + \hat{k}_2^n x^2\phi_n'' + \hat{k}_3^n\phi_n + \hat{k}_4^n x^2\phi_n, k_1^m\phi_m'' + k_2^m x^2\phi_m'' + k_3^m\phi_m + k_4^m x^2\phi_m\,\Big)_{L^2}$$

$$= \begin{bmatrix} \hat{k}_1^m \\ \hat{k}_2^m \\ \hat{k}_3^m \\ \hat{k}_4^m \end{bmatrix}^T \begin{bmatrix} I & [B_{10}]_{mn} & [B_2]_{mn} & [B_3]_{mn} \\ [B_{10}]_{mn} & [B_{11}]_{mn} & [B_3]_{mn} & [B_{12}]_{mn} \\ [B_2]_{mn} & [B_3]_{mn}^T & [B_1]_{mn} & [B_4]_{mn}^T \\ [B_3]_{mn}^T & [B_{12}]_{mn}^T & [B_4]_{mn} & [B_7]_{mn} \end{bmatrix} \begin{bmatrix} k_1^n \\ k_2^n \\ k_3^n \\ k_4^n \end{bmatrix}.$$

INTLAB code (for confirmation)

"CHECKING_E/main.m"

```
format long;
intvalinit('DisplayInfsup');
intvalinit('SharpIVmult');


% -=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=
%     setting parameters
% -=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=
K = 200;
zv = intval(zeros(K,1));  Id = intval(eye(K));  zm = intval(zeros(K));
su = intval('5');
sr = intval('1')/5000;
sl = intval('1')/1000;
sm = intval('1')/2000;
t = 10;
ur0 = zeros(K,1); ur0(1) = t; ui0 = ur0; vr0 = ur0; vi0 = ur0;



% -=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=
%     reading matrices and approximate solution
% -=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=
load('../B.mat');
[a,R,lambda,mu,ur,ui,vr,vi] = input_approximate_solution(K);


[C11,C12,C13,C14,C22,C23,C24,C33,C34,C44] = generate_C(K,a,R,lambda,mu,...
                                       B1,B2,B3,B4,B7,B10,B11,B12);

w = importdata('w.dat');
urs = w(1:K);        uis = w(K+1:2*K);
vrs = w(2*K+1:3*K);  vis = w(3*K+1:4*K);
as = w(4*K+1);       Rs = w(4*K+2);
lambdas = w(4*K+3);  mus = w(4*K+4);


%
% ----/ norm0 /---------------------------- %
%
ur0 = intval(ur0);  ui0 = intval(ui0);  vr0 = intval(vr0);  vi0 = intval(vi0);
M = [ur0*ur0',zm,zm,zm,-ur0,zv,zv,zv;
     zm,ui0*ui0',zm,zm,zv,-ui0,zv,zv;
     zm,zm,vr0*vr0',zm,zv,zv,-vr0,zv;
     zm,zm,zm,vi0*vi0',zv,zv,zv,-vi0;
     -ur0',zv',zv',zv',1,0,0,0;
     zv',-ui0',zv',zv',0,1,0,0;
     zv',zv',-vr0',zv',0,0,1,0;
     zv',zv',zv',-vi0',0,0,0,1];
norm0 = w'*M*w;

%
% ----/ norm1 /---------------------------- %
%
t11 = inner_product(1,1,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
t12 = inner_product(1,2,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
t13 = inner_product(1,3,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
t22 = inner_product(2,2,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
t23 = inner_product(2,3,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
t33 = inner_product(3,3,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
z11 = vectors_z(1,1,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
z12 = vectors_z(2,1,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
z13 = vectors_z(3,1,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
z21 = vectors_z(1,2,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
z22 = vectors_z(2,2,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
z23 = vectors_z(3,2,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);

M =[C11, -C12',  zm, zm,  z11,  z12,  z13, zv;
   -C12,   C22,  zm, zm  -z21, -z22, -z23, zv;
    zm,     zm,  zm, zm,   zv,   zv,   zv, zv;
    zm,     zm,  zm, zm,   zv,   zv,   zv, zv;
   z11', -z21', zv', zv', t11,  t12,  t13,  0;
   z12', -z22', zv', zv', t12,  t22,  t23,  0;
   z13', -z23', zv', zv', t13,  t23,  t33,  0;
    zv',   zv', zv', zv',   0,    0,    0,  0];
norm1 = w'*M*w;
```

"CHECKING_E/main.m"

```
%
% ----/ norm2 /---------------------------- %
%
t44 = inner_product(4,4,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
t45 = inner_product(4,5,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
t46 = inner_product(4,6,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
t55 = inner_product(5,5,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
t56 = inner_product(5,6,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
t66 = inner_product(6,6,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
z24 = vectors_z(4,2,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
z14 = vectors_z(4,1,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
z25 = vectors_z(5,2,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
z15 = vectors_z(5,1,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
z26 = vectors_z(6,2,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
z16 = vectors_z(6,1,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
M =[C22,   C12,  zm, zm,  z24,  z25,  z26, zv;
   C12',   C11,  zm, zm   z14,  z15,  z16, zv;
    zm,     zm,  zm, zm,   zv,   zv,   zv, zv;
    zm,     zm,  zm, zm,   zv,   zv,   zv, zv;
   z24',  z14', zv', zv', t44,  t45,  t46,  0;
   z25',  z15', zv', zv', t45,  t55,  t56,  0;
   z26',  z16', zv', zv', t46,  t56,  t66,  0;
    zv',   zv', zv', zv',   0,    0,    0,  0];
norm2 = w'*M*w;

%
% ----/ norm3 /---------------------------- %
%
t77 = inner_product(7,7,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
t78 = inner_product(7,8,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
t79 = inner_product(7,9,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
t710 = inner_product(7,10,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
t88 = inner_product(8,8,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
t89 = inner_product(8,9,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
t810 = inner_product(8,10,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
t99 = inner_product(9,9,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
t910 = inner_product(9,10,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
t1010 = inner_product(10,10,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
z47 = vectors_z(7,4,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
z48 = vectors_z(8,4,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
z49 = vectors_z(9,4,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
z410 = vectors_z(10,4,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
z37 = vectors_z(7,3,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
z38 = vectors_z(8,3,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
z39 = vectors_z(9,3,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
z310 = vectors_z(10,3,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
z17 = vectors_z(7,1,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
z18 = vectors_z(8,1,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
z19 = vectors_z(9,1,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
z110 = vectors_z(10,1,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
z27 = vectors_z(7,2,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
z28 = vectors_z(8,2,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
z29 = vectors_z(9,2,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
z210 = vectors_z(10,2,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
M =[C44,   -C34,   C14,   -C24,  z47,  z48,  z49,  z410;
  -C34',    C33,  -C13,    C23, -z37, -z38, -z39, -z310;
   C14',  -C13',   C11,  -C12',  z17,  z18,  z19,  z110;
  -C24',   C23',  -C12,    C22, -z27, -z28, -z29, -z210;
   z47',  -z37',  z17',  -z27',  t77,  t78,  t79,  t710;
   z48',  -z38',  z18',  -z28',  t78,  t88,  t89,  t810;
   z49',  -z39',  z19',  -z29',  t79,  t89,  t99,  t910;
  z410', -z310', z110', -z210', t710, t810, t910, t1010];
norm3 = w'*M*w;


%
% ----/ norm4 /---------------------------- %
%
t1111 = inner_product(11,11,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
t1112 = inner_product(11,12,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
t1113 = inner_product(11,13,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
t1114 = inner_product(11,14,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
t1212 = inner_product(12,12,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
t1213 = inner_product(12,13,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
t1214 = inner_product(12,14,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
t1313 = inner_product(13,13,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
t1314 = inner_product(13,14,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
t1414 = inner_product(14,14,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
```

"CHECKING_E/main.m"

```
z311 = vectors_z(11,3,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
z312 = vectors_z(12,3,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
z313 = vectors_z(13,3,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
z314 = vectors_z(14,3,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
z411 = vectors_z(11,4,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
z412 = vectors_z(12,4,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
z413 = vectors_z(13,4,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
z414 = vectors_z(14,4,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
z211 = vectors_z(11,2,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
z212 = vectors_z(12,2,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
z213 = vectors_z(13,2,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
z214 = vectors_z(14,2,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
z111 = vectors_z(11,1,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
z112 = vectors_z(12,1,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
z113 = vectors_z(13,1,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
z114 = vectors_z(14,1,a,R,lambda,mu,ur,ui,vr,vi,B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm);
M =[C33,   C34',   C23,   C13,  z311,   z312,  z313,  z314;
    C34,    C44,   C24,   C14,  z411,   z412,  z413,  z414;
   C23',   C24',   C22,   C12,  z211,   z212,  z213,  z214;
   C13',   C14',  C12',   C11,  z111,   z112,  z113,  z114;
  z311',  z411', z211', z111',  t1111, t1112, t1113, t1114;
  z312',  z412', z212', z112',  t1112, t1212, t1213, t1214;
  z313',  z413', z213', z113',  t1113, t1213, t1313, t1314;
  z314',  z414', z214', z114',  t1114, t1214, t1314, t1414];
norm4 = w'*M*w;


%
% ----/ E /---------------------------- %
%
ur0 = intval(ur0);  ui0 = intval(ui0);  vr0 = intval(vr0);  vi0 = intval(vi0);
E11 = C11 + C22 + C33 + C44 + ur0*ur0';
E12 = -C12' + C12 - C34 + C34';
E13 = C14 + C23;
E14 = -C24 + C13;
E22 = C11 + C22 + C33 + C44 + ui0*ui0';
E23 = -C13 + C24;
E24 = C14 + C23;
E33 =  C11 + C22 + vr0*vr0';
E34 = -C12' + C12;
E44 =  C11 + C22 + vi0*vi0';
E15 = z11 + z24 + z47 + z311 - ur0;
E16 = z12 + z25 + z48 + z312;
E17 = z13 + z26 + z49 + z313;
E18 = z410 + z314;
E25 = -z21 + z14 - z37 + z411;
E26 = -z22 + z15 - z38 + z412 -ui0;
E27 = -z23 + z16 - z39 + z413;
E28 = -z310 + z414;
E35 = z17 + z211;
E36 = z18 + z212;
E37 = z19 + z213 - vr0;
E38 = z110 + z214;
E45 = -z27 + z111;
E46 = -z28 + z112;
E47 = -z29 + z113;
E48 = -z210 + z114 -vi0;
E55 = t11 + t44 + t77 + t1111 + 1;
E56 = t12 + t45 + t78 + t1112;
E57 = t13 + t46 + t79 + t1113;
E58 = t710 + t1114;
E66 = t22 + t55 + t88 + t1212 + 1;
E67 = t23 + t56 + t89 + t1213;
E68 = t810 + t1214;
E77 = t33 + t66 + t99 + t1313 + 1;
E78 = t910 + t1314;
E88 = t1010 + t1414 + 1;

E = [ E11,  E12,  E13,  E14,  E15,  E16,  E17,  E18;
      E12', E22,  E23,  E24,  E25,  E26,  E27,  E28;
      E13', E23', E33,  E34,  E35,  E36,  E37,  E38;
      E14', E24', E34', E44,  E45,  E46,  E47,  E48;
      E15', E25', E35', E45', E55,  E56,  E57,  E58;
      E16', E26', E36', E46', E56', E66,  E67,  E68;
      E17', E27', E37', E47', E57', E67', E77,  E78;
      E18', E28', E38', E48', E58', E68', E78', E88];

norm5 = w'*E*w;
```

"CHECKING_E/Generate_C.m"

```
function [C11,C12,C13,C14,C22,C23,C24,C33,C34,C44] = ...
  generate_C(K,a,R,lambda,mu,B1,B2,B3,B4,B7,B10,B11,B12);

Id = intval(eye(K));

% C11
k1 = 2*a^2; k2 = 0; k3 = -a^4; k4 = 0;
j1 = 2*a^2; j2 = 0; j3 = -a^4; j4 = 0;
C11 = k1*(j1*Id  + j2*B10  + j3*B2 + j4*B3 ) + ...
      k2*(j1*B10 + j2*B11  + j3*B3 + j4*B12) + ...
      k3*(j1*B2  + j2*B3'  + j3*B1 + j4*B4') + ...
      k4*(j1*B3' + j2*B12' + j3*B4 + j4*B7 );

% C12
k1 = 2*a^2; k2 = 0; k3 = -a^4; k4 = 0;
j1 = a*R-lambda; j2 = -a*R; j3 = a*(-R*(a^2-2)+a*lambda); j4 = a^3*R;
C12 = k1*(j1*Id  + j2*B10  + j3*B2 + j4*B3 ) + ...
      k2*(j1*B10'+ j2*B11  + j3*B3 + j4*B12) + ...
      k3*(j1*B2' + j2*B3'  + j3*B1 + j4*B4') + ...
      k4*(j1*B3' + j2*B12' + j3*B4 + j4*B7 );

% C13
k1 = 2*a^2; k2 = 0; k3 = -a^4; k4 = 0;
j1 = R-mu; j2 = -R; j3 =-R*(3*a^2-2)+a*(a*mu+2*lambda); j4 = 3*a^2*R;
C13 = k1*(j1*Id  + j2*B10  + j3*B2 + j4*B3 ) + ...
      k2*(j1*B10'+ j2*B11  + j3*B3 + j4*B12) + ...
      k3*(j1*B2' + j2*B3'  + j3*B1 + j4*B4') + ...
      k4*(j1*B3' + j2*B12' + j3*B4 + j4*B7 );

% C14
k1 = 2*a^2; k2 = 0; k3 = -a^4; k4 = 0;
j1 = 4*a; j2 = 0; j3 = -4*a^3; j4 = 0;
C14 = k1*(j1*Id  + j2*B10  + j3*B2 + j4*B3 ) + ...
      k2*(j1*B10'+ j2*B11  + j3*B3 + j4*B12) + ...
      k3*(j1*B2' + j2*B3'  + j3*B1 + j4*B4') + ...
      k4*(j1*B3' + j2*B12' + j3*B4 + j4*B7 );

% C22
k1 = a*R-lambda; k2 = -a*R; k3 = a*(-R*(a^2-2)+a*lambda); k4 = a^3*R;
j1 = a*R-lambda; j2 = -a*R; j3 = a*(-R*(a^2-2)+a*lambda); j4 = a^3*R;
C22 = k1*(j1*Id  + j2*B10  + j3*B2 + j4*B3 ) + ...
      k2*(j1*B10 + j2*B11  + j3*B3 + j4*B12) + ...
      k3*(j1*B2  + j2*B3'  + j3*B1 + j4*B4') + ...
      k4*(j1*B3' + j2*B12' + j3*B4 + j4*B7 );

% C23
k1 = a*R-lambda; k2 = -a*R; k3 = a*(-R*(a^2-2)+a*lambda); k4 = a^3*R;
j1 = R-mu; j2 = -R; j3 =-R*(3*a^2-2)+a*(a*mu+2*lambda); j4 = 3*a^2*R;
C23 = k1*(j1*Id  + j2*B10  + j3*B2 + j4*B3 ) + ...
      k2*(j1*B10 + j2*B11  + j3*B3 + j4*B12) + ...
      k3*(j1*B2  + j2*B3'  + j3*B1 + j4*B4') + ...
      k4*(j1*B3' + j2*B12' + j3*B4 + j4*B7 );

% C24
k1 = a*R-lambda; k2 = -a*R; k3 = a*(-R*(a^2-2)+a*lambda); k4 = a^3*R;
j1 = 4*a; j2 = 0; j3 = -4*a^3; j4 = 0;
C24 = k1*(j1*Id  + j2*B10  + j3*B2 + j4*B3 ) + ...
      k2*(j1*B10 + j2*B11  + j3*B3 + j4*B12) + ...
      k3*(j1*B2  + j2*B3'  + j3*B1 + j4*B4') + ...
      k4*(j1*B3' + j2*B12' + j3*B4 + j4*B7 );

% C33
k1 = R-mu; k2 = -R; k3 =-R*(3*a^2-2)+a*(a*mu+2*lambda); k4 = 3*a^2*R;
j1 = R-mu; j2 = -R; j3 =-R*(3*a^2-2)+a*(a*mu+2*lambda); j4 = 3*a^2*R;
C33 = k1*(j1*Id  + j2*B10  + j3*B2 + j4*B3 ) + ...
      k2*(j1*B10 + j2*B11  + j3*B3 + j4*B12) + ...
      k3*(j1*B2  + j2*B3'  + j3*B1 + j4*B4') + ...
      k4*(j1*B3' + j2*B12' + j3*B4 + j4*B7 );
```

```
% C34
k1 = R-mu; k2 = -R; k3 =-R*(3*a^2-2)+a*(a*mu+2*lambda); k4 = 3*a^2*R;
j1 = 4*a; j2 = 0; j3 = -4*a^3; j4 = 0;
C34 = k1*(j1*Id  + j2*B10  + j3*B2 + j4*B3 ) + ...
      k2*(j1*B10 + j2*B11  + j3*B3 + j4*B12) + ...
      k3*(j1*B2  + j2*B3'  + j3*B1 + j4*B4') + ...
      k4*(j1*B3' + j2*B12' + j3*B4 + j4*B7 );

% C44
k1 = 4*a; k2 = 0; k3 = -4*a^3; k4 = 0;
j1 = 4*a; j2 = 0; j3 = -4*a^3; j4 = 0;
C44 = k1*(j1*Id  + j2*B10  + j3*B2 + j4*B3 ) + ...
      k2*(j1*B10 + j2*B11  + j3*B3 + j4*B12) + ...
      k3*(j1*B2  + j2*B3'  + j3*B1 + j4*B4') + ...
      k4*(j1*B3' + j2*B12' + j3*B4 + j4*B7 );
```

“CHECKING_E/Generate_C.m”

```
function s = inner_product(k,l,a,R,lambda,mu,ur,ui,vr,vi, ...
B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm)
  %      [talbe for z_k]
  %  -------------------------------------
  %       ur   ui   vr   vi      list
  %  -------------------------------------
  %  z1   y1  -y2    0    0  [1,-2,0,0]
  %  z2    0  -y3    0    0  [0,-3,0,0]
  %  z3    0  -y4    0    0  [0,-4,0,0]
  %  z4   y2   y1    0    0  [2,1,0,0]
  %  z5   y3    0    0    0  [3,0,0,0]
  %  z6   y4    0    0    0  [4,0,0,0]
  %  z7   y5  -y6   y1  -y2  [5,-6,1,-2]
  %  z8    0  -y7    0  -y3  [0,-7,0,-3]
  %  z9    0  -y8    0  -y4  [0,-8,0,-4]
  %  z10   0  -y9    0    0  [0,-9,0,0]
  %  z11  y6   y5   y2   y1  [6,5,2,1]
  %  z12  y7    0   y3    0  [7,0,3,0]
  %  z13  y8    0   y4    0  [8,0,4,0]
  %  z14  y9    0    0    0  [9,0,0,0]
  %  -------------------------------------

% select z's
[w,left] = generate_z(k,ur,ui,vr,vi);
[x,right] = generate_z(l,ur,ui,vr,vi);
Id = eye(200);

% compute inner-product
s = 0;
for i = 1:4
  k1 = coefficient_k(left(i),a,R,lambda,mu,su,sr,sl,sm);
  for j = 1:4
    k2 = coefficient_k(right(j),a,R,lambda,mu,su,sr,sl,sm);
    D = k1(1)*(k2(1)*Id  + k2(2)*B10  + k2(3)*B2 + k2(4)*B3 ) + ...
        k1(2)*(k2(1)*B10 + k2(2)*B11  + k2(3)*B3 + k2(4)*B12) + ...
        k1(3)*(k2(1)*B2  + k2(2)*B3'  + k2(3)*B1 + k2(4)*B4') + ...
        k1(4)*(k2(1)*B3' + k2(2)*B12' + k2(3)*B4 + k2(4)*B7 );
    s = s + x(:,j)'*D*w(:,i);
  end
end
```

“CHECKING_E/coefficient_k.m”

```
function k = coefficient_k(j,a,R,lambda,mu,su,sr,sl,sm)
  u = intval('1');  k = intval(zeros(4,1));
  switch(j)
    case 1
      k(1) = 4*a*su;  k(3) = -4*a^3*su;
    case 2
    k(1) = R*su;  k(2) = -R*su;  k(3) = (-R*(3*a^2-2)+2*a*lambda)*su;
    k(4) = 3*a^2*R*su;
    case 3
      k(1) = a*su/sr;  k(2) = -a*su/sr;  k(3) = a*(-a^2+2)*su/sr;
      k(4) = a^3*su/sr;
    case 4
      k(1) = -u*su/sl;  k(3) = a^2*su/sl;
    case 5
      k(1) = 4*u*su;  k(3) = -12*a^2*su;
    case 6
      k(3) = 2*(a*(-3*R+mu)+lambda)*su;  k(4) = 6*a*R*su;
    case 7
      k(1) = u*su/sr;  k(2) = -u*su/sr;  k(3) = (-3*a^2+2)*su/sr;
      k(4) = 3*a^2*su/sr;
    case 8
      k(3) = 2*a*su/sl;
    case 9
      k(1) = -u*su/sm;  k(3) = a^2*su/sm;
  end
```

“CHECKING_E/generate_z.m”

```
function [w,list] = generate_z(j,ur,ui,vr,vi)
  w = intval(zeros(200,4));
  switch(j)
    case 1
      w(:,1) = ur;  w(:,2) = -ui;
      list = [1,2,0,0];
    case 2
      w(:,2) = -ui;
      list = [0,3,0,0];
    case 3
      w(:,2) = -ui;
      list = [0,4,0,0];
    case 4
      w(:,1) = ur;  w(:,2) = ui;
      list = [2,1,0,0];
    case 5
      w(:,1) = ur;
      list = [3,0,0,0];
    case 6
      w(:,1) = ur;
      list = [4,0,0,0];
    case 7
      w(:,1) = ur;  w(:,2) =- ui;  w(:,3) = vr;  w(:,4) = -vi;
      list = [5,6,1,2];
    case 8
      w(:,2) = -ui;  w(:,4) = -vi;
      list = [0,7,0,3];
    case 9
      w(:,2) = -ui;  w(:,4) = -vi;
      list = [0,8,0,4];
    case 10
      w(:,2) = -ui;
      list = [0,9,0,0];
    case 11
      w(:,1) = ur;  w(:,2) = ui;  w(:,3) = vr;  w(:,4) = vi;
      list = [6,5,2,1];
    case 12
      w(:,1) = ur;  w(:,3) = vr;
      list = [7,0,3,0];
    case 13
      w(:,1) = ur;  w(:,3) = vr;
      list = [8,0,4,0];
    case 14
      w(:,1) = ur;
      list = [9,0,0,0];
  end
```

```
"CHECKING_E/vectors_z.m"
--------------------------------------------------------------------------------
function v = vectors_z(k,l,a,R,lambda,mu,ur,ui,vr,vi, ...
  B1,B2,B3,B4,B7,B10,B11,B12,su,sr,sl,sm)

k2 = intval(zeros(4,1));
switch(l)
 case(1)
   k2(1) = 2*a^2; k2(2) = 0; k2(3) = -a^4; k2(4) = 0;
 case(2)
   k2(1) = a*R-lambda; k2(2) = -a*R; k2(3) = a*(-R*(a^2-2)+a*lambda);
   k2(4) = a^3*R;
 case(3)
   k2(1) = R-mu; k2(2) = -R; k2(3) =-R*(3*a^2-2)+a*(a*mu+2*lambda);
   k2(4) = 3*a^2*R;
 case(4)
   k2(1) = 4*a; k2(2) = 0; k2(3) = -4*a^3; k2(4) = 0;
end

[w,left] = generate_z(k,ur,ui,vr,vi);

Id = eye(200); v = intval(zeros(200,1));
for i = 1:4
  k1 = coefficient_k(left(i),a,R,lambda,mu,su,sr,sl,sm);
  F = k1(1)*(k2(1)*Id  + k2(2)*B10  + k2(3)*B2 + k2(4)*B3 ) + ...
      k1(2)*(k2(1)*B10 + k2(2)*B11  + k2(3)*B3 + k2(4)*B12) + ...
      k1(3)*(k2(1)*B2  + k2(2)*B3'  + k2(3)*B1 + k2(4)*B4') + ...
      k1(4)*(k2(1)*B3' + k2(2)*B12' + k2(3)*B4 + k2(4)*B7 );
  v = v + F*w(:,i);
end
```

2.4 Mathematica code (for confirmation

```
"CHECKING_E/main.ma"
--------------------------------------------------------------------------------
(* defining Ledgendre polynomial *)
  phi[n_]:= (-1)^(2n) Sqrt[4n+1]/(2n)! /2^(2n+1/2)D[(1-x^2)^(2n),{x,2n-2}];

(* setting scaling parameters *)
  su = 5;
  sr = 1/5000;
  sl = 1/1000;
  sm = 1/2000;

(* reading approximate solution as rational number from file *)
  K = 100;
  w = ReadList["../APPROXIMATE_SOLUTION/app_rational.dat"];
  s = 0; Do[ s = s + phi[i] w[[i]], {i,K}];      ur = Expand[s];
  s = 0; Do[ s = s + phi[i] w[[K+i]], {i,K}];    ui = Expand[s];
  s = 0; Do[ s = s + phi[i] w[[2K+i]], {i,K}];   vr = Expand[s];
  s = 0; Do[ s = s + phi[i] w[[3K+i]], {i,K}];   vi = Expand[s];
  a = w[[4K+1]];
  R = w[[4K+2]];
  lambda = w[[4K+3]];
  mu = w[[4K+4]];
```

```
(* setting z1 - z14 excluding s_u *)

z1 = su( 4 a D[ur,{x,2}] - 4 a^3 ur - R D[ui,{x,2}] + R x^2 D[ui,{x,2}] -
     ( -R(3 a^2 - 2) + 2 a lambda )ui - 3 a^2 R x^2 ui );

z2 = su( - a D[ui,{x,2}] + a x^2 D[ui,{x,2}] - a(-a^2+2) ui - a^3 x^2 ui)/sr;

z3 = su( D[ui,{x,2}] - a^2 ui)/sl;

z4 = su( R D[ur,{x,2}] - R x^2 D[ur,{x,2}] + ( -R(3 a^2 - 2) + 2 a lambda )ur+
     3 a^2 R x^2 ur + 4 a D[ui,{x,2}] - 4 a^3 ui) ;

z5 = su( a D[ur,{x,2}] - a x^2 D[ur,{x,2}] + a(-a^2+2) ur + a^3 x^2 ur)/sr;

z6 = su( - D[ur,{x,2}] + a^2 ur)/sl;

z7 = su( 4 D[ur,{x,2}] -12 a^2 ur -2(a(-3 R + mu) + lambda)ui - 6 a R x^2 ui +
      4 a D[vr,{x,2}] - 4 a^3 vr - R D[vi,{x,2}] + R x^2 D[vi,{x,2}] -
      ( -R(3 a^2 - 2) + 2 a lambda )vi - 3 a^2 R x^2 vi);

z8 = su( - D[ui,{x,2}] + x^2 D[ui,{x,2}] - (-3 a^2 + 2)ui - 3 a^2 x^2 ui -
     a D[vi,{x,2}] + a x^2 D[vi,{x,2}] - a(-a^2+2) vi - a^3 x^2 vi)/sr;

z9 = su( - 2 a ui + D[vi,{x,2}] - a^2 vi)/sl;

z10 = su(D[ui,{x,2}] - a^2 ui)/sm;

z11 = su( 2(a(-3 R + mu) + lambda)ur + 6 a R x^2 ur + 4 D[ui,{x,2}] -12 a^2 ui+
      R D[vr,{x,2}] - R x^2 D[vr,{x,2}] + ( -R(3 a^2 - 2) + 2 a lambda )vr +
      3 a^2 R x^2 vr + 4 a D[vi,{x,2}] - 4 a^3 vi);

z12 = su( D[ur,{x,2}] - x^2 D[ur,{x,2}] + (-3 a^2 + 2)ur + 3 a^2 x^2 ur +
     a D[vr,{x,2}] - a x^2 D[vr,{x,2}] + a(-a^2+2) vr + a^3 x^2 vr)/sr;

z13 = su( 2 a ur - D[vr,{x,2}] + a^2 vr)/sl;

z14 = su( -D[ur,{x,2}] + a^2 ur)/sm;

z1 = N[z1,500];
z2 = N[z2,500];
z3 = N[z3,500];
z4 = N[z4,500];
z5 = N[z5,500];
z6 = N[z6,500];
z7 = N[z7,500];
z8 = N[z8,500];
z9 = N[z9,500];
z10 = N[z10,500];
z11 = N[z11,500];
z12 = N[z12,500];
z13 = N[z13,500];
z14 = N[z14,500];

(* generate [ur*,ui*,vr*,vi*,a*,R*,lambda*,mu*] by random number *)
  K = 200;
(*  w = RandomReal[{-1,1},4K+4,WorkingPrecision -> 500]; *)
  w = Import["w.dat"];
(*  w = << w.dat; *)
  s = 0; Do[ s = s + phi[i] w[[i]], {i,K}];     urs = Expand[s];
  s = 0; Do[ s = s + phi[i] w[[K+i]], {i,K}];   uis = Expand[s];
  s = 0; Do[ s = s + phi[i] w[[2K+i]], {i,K}];  vrs = Expand[s];
  s = 0; Do[ s = s + phi[i] w[[3K+i]], {i,K}];  vis = Expand[s];
```

```
  as = w[[4K+1]];
  Rs = w[[4K+2]];
  lambdas = w[[4K+3]];
  mus = w[[4K+4]];


(* norm 0 *)
  u0 = 10 phi[1];
  tur = Integrate[Expand[D[urs,{x,2}]D[u0,{x,2}]],{x,-1,1}];
  tui = Integrate[Expand[D[uis,{x,2}]D[u0,{x,2}]],{x,-1,1}];
  tvr = Integrate[Expand[D[vrs,{x,2}]D[u0,{x,2}]],{x,-1,1}];
  tvi = Integrate[Expand[D[vis,{x,2}]D[u0,{x,2}]],{x,-1,1}];
norm0 = ( as - tur )^2 + ( Rs - tui )^2 + (lambdas - tvr)^2 +( mus - tvi )^2;


(* norm 1 *)
k1 = 2*a^2; k2 = 0; k3 = -a^4; k4 = 0; (* q1 *)
p1 = k1 D[urs,{x,2}] + k2 x^2 D[urs,{x,2}] + k3 urs + k4 x^2 urs;
k1 = a*R-lambda; k2 = -a*R; k3 = a*(-R*(a^2-2)+a*lambda); k4 = a^3*R; (* q2 *)
p2 = k1 D[uis,{x,2}] + k2 x^2 D[uis,{x,2}] + k3 uis + k4 x^2 uis;
s = Expand[z1 as + z2 Rs + z3 lambdas + p1 - p2];
norm1 = Integrate[Expand[s^2],{x,-1,1}];



(* norm 2 *)
k1 = a*R-lambda; k2 = -a*R; k3 = a*(-R*(a^2-2)+a*lambda); k4 = a^3*R; (* q2 *)
p1 = k1 D[urs,{x,2}] + k2 x^2 D[urs,{x,2}] + k3 urs + k4 x^2 urs;
k1 = 2*a^2; k2 = 0; k3 = -a^4; k4 = 0; (* q1 *)
p2 = k1 D[uis,{x,2}] + k2 x^2 D[uis,{x,2}] + k3 uis + k4 x^2 uis;
s = Expand[z4 as + z5 Rs + z6 lambdas + p1 + p2];
norm2 = Integrate[Expand[s^2],{x,-1,1}];


(* norm 3 *)
k1 = 4*a; k2 = 0; k3 = -4*a^3; k4 = 0;  (* q4 *)
p1 = k1 D[urs,{x,2}] + k2 x^2 D[urs,{x,2}] + k3 urs + k4 x^2 urs;
k1 = R-mu; k2 = -R; k3 =-R*(3*a^2-2)+a*(a*mu+2*lambda); k4 = 3*a^2*R; (* q3 *)
p2 = k1 D[uis,{x,2}] + k2 x^2 D[uis,{x,2}] + k3 uis + k4 x^2 uis;
k1 = 2*a^2; k2 = 0; k3 = -a^4; k4 = 0; (* q1 *)
p3 = k1 D[vrs,{x,2}] + k2 x^2 D[vrs,{x,2}] + k3 vrs + k4 x^2 vrs;
k1 = a*R-lambda; k2 = -a*R; k3 = a*(-R*(a^2-2)+a*lambda); k4 = a^3*R; (* q2 *)
p4 = k1 D[vis,{x,2}] + k2 x^2 D[vis,{x,2}] + k3 vis + k4 x^2 vis;
s = Expand[z7 as + z8 Rs + z9 lambdas + z10 mus + p1 - p2 + p3 -p4];
norm3 = Integrate[Expand[s^2],{x,-1,1}];


(* norm 4 *)
k1 = R-mu; k2 = -R; k3 =-R*(3*a^2-2)+a*(a*mu+2*lambda); k4 = 3*a^2*R; (* q3 *)
p1 = k1 D[urs,{x,2}] + k2 x^2 D[urs,{x,2}] + k3 urs + k4 x^2 urs;
k1 = 4*a; k2 = 0; k3 = -4*a^3; k4 = 0;  (* q4 *)
p2 = k1 D[uis,{x,2}] + k2 x^2 D[uis,{x,2}] + k3 uis + k4 x^2 uis;
k1 = a*R-lambda; k2 = -a*R; k3 = a*(-R*(a^2-2)+a*lambda); k4 = a^3*R; (* q2 *)
p3 = k1 D[vrs,{x,2}] + k2 x^2 D[vrs,{x,2}] + k3 vrs + k4 x^2 vrs;
k1 = 2*a^2; k2 = 0; k3 = -a^4; k4 = 0; (* q1 *)
p4 = k1 D[vis,{x,2}] + k2 x^2 D[vis,{x,2}] + k3 vis + k4 x^2 vis;
s = Expand[z11 as + z12 Rs + z13 lambdas + z14 mus + p1 + p2 + p3 + p4];
norm4 = Integrate[Expand[s^2],{x,-1,1}];
```

2.5 Confirmation of matrix E and related norms

We generate $\boldsymbol{w} \in \mathbb{R}^{4N+4}$ as random number by Mathematica and compute each norms. Each MATLAB and Mathematica read the data in “`w.dat`”.

$$(a - \left(u_r, u_r^0\right)_{H_0^2})^2 + (R - \left(u_i, u_i^0\right)_{H_0^2})^2 + (\lambda_i - \left(v_r, v_r^0\right)_{H_0^2})^2 + (\mu_i - \left(v_i, v_i^0\right)_{H_0^2})^2$$
$$= 1.6840241755052_5^6 \times 10^2 \ni 1.684024175505254\cdots \times 10^2 \text{ (Mathematica)}$$

$$\|z_1 a + z_2 R + z_3 \lambda_i + \hat{q}_1 u_r - \hat{q}_2 u_i\|^2$$
$$= 4.4160282924427_{21}^{35} \times 10^8 \ni 4.41602829242727\cdots \times 10^8 \text{ (Mathematica)}$$

$$\|z_4 a + z_5 R + z_6 \lambda_i + \hat{q}_2 u_r + \hat{q}_1 u_i\|^2$$
$$= 4.1296563879684_{28}^{43} \times 10^8 \ni 4.12965638796835 \times 10^8 \text{ (Mathematica)}$$

$$\|z_7 a + z_8 R + z_9 \lambda_i + z_{10}\mu_i + \hat{q}_4 u_r - \hat{q}_3 u_i + \hat{q}_1 v_r - \hat{q}_2 v_i\|^2$$
$$= 8.7789623474676_{50}^{88} \times 10^8 \ni 8.77896234746768\cdots \times 10^8 \text{ (Mathematica)}$$

$$\|z_{11} a + z_{12} R + z_{13} \lambda_i + z_{14}\mu_i + \hat{q}_3 u_r + \hat{q}_4 u_i + \hat{q}_2 v_r + \hat{q}_1 v_i\|^2$$
$$= 7.90223672374_{556}^{600} \times 10^8 \ni 7.90223672374577\cdots \times 10^8 \text{ (Mathematica)}$$

and finally

$$\|qw\|_Y = 50226.37298833_{42}^{54} \ni 50226.3729883347\cdots \text{ (Mathematica).}$$

2.6 Upper bound of ν_2

When

$$s_u = 5, \quad s_r = \frac{1}{5000}, \quad s_l = \frac{1}{1000}, \quad s_m = \frac{1}{2000},$$

we obtain

$$\sqrt{\|E\|_2} = 6466.{}_{58}^{62}$$

because of

```
>> sqrt(norm(E))
intval ans =
  1.0e+003 *
[   6.46658513217555,   6.46661757177779]
```

3 Bound of ν_3

For $\nu_3 > 0$ satisfying

$$\|q(I-P_h)w\|_Y \le \nu_3 \|(I-P_h)w\|_X, \qquad \forall w \in X,$$

we take

$$w_* := (I-P_h)w = [u_r^\perp, u_i^\perp, v_r^\perp, v_i^\perp, 0,0,0,0]^T \in (S_h^\perp)^4 \times \mathbb{R}^4$$

for each $w \in X$. Then from (6), we have

$$\begin{aligned}\|qw_*\|_Y^2 &= \|\hat{q}_1 u_r^\perp - \hat{q}_2 u_i^\perp\|^2 + \|\hat{q}_2 u_r^\perp + \hat{q}_1 u_i^\perp\|^2 \\ &\quad + \|\hat{q}_4 u_r^\perp - \hat{q}_3 u_i^\perp + \hat{q}_1 v_r^\perp - \hat{q}_2 v_i^\perp\|^2 + \|\hat{q}_3 u_r^\perp + \hat{q}_4 u_i^\perp + \hat{q}_2 v_r^\perp + \hat{q}_1 v_i^\perp\|^2.\end{aligned}$$

Again, for $u \in \{u_r^\perp, u_i^\perp, v_r^\perp, v_i^\perp\}$, so that

$$\hat{q}_1 u = 2\hat{a}^2 D^2 u - \hat{a}^4 u,$$

$$\hat{q}_2 u = \left(\hat{a}\hat{R} - \hat{\lambda}_i - \hat{a}\hat{R}x^2\right) D^2 u + \left(\hat{a}(-\hat{R}(\hat{a}^2-2) + \hat{a}\hat{\lambda}_i) + \hat{a}^3\hat{R}x^2\right) u,$$

$$\hat{q}_3 u = \left(\hat{R} - \hat{\mu}_i - \hat{R}x^2\right) D^2 u + \left(-\hat{R}(3\hat{a}^2-2) + \hat{a}(\hat{a}\hat{\mu}_i + 2\hat{\lambda}_i) + 3\hat{a}^2\hat{R}x^2\right) u,$$

$$\hat{q}_4 u = 4\hat{a} D^2 u - 4\hat{a}^3 u,$$

and

$$\begin{aligned}\left\|\hat{a}\hat{R} - \hat{\lambda}_i - \hat{a}\hat{R}x^2\right\|_\infty &= \hat{a}\hat{R} - \hat{\lambda}_i, \\ \left\|\hat{a}(-\hat{R}(\hat{a}^2-2) + \hat{a}\hat{\lambda}_i) + \hat{a}^3\hat{R}x^2\right\|_\infty &= 2\hat{a}\hat{R} + \hat{a}^2\hat{\lambda}_i, \\ \left\|\hat{R} - \hat{\mu}_i - \hat{R}x^2\right\|_\infty &= \hat{R} - \hat{\mu}_i, \\ \left\|-3\hat{a}^2\hat{R} + 2\hat{R} + \hat{a}(\hat{a}\hat{\mu}_i + 2\hat{\lambda}_i) + 3\hat{a}^2\hat{R}x^2\right\|_\infty &= 2\hat{R} + \hat{a}(\hat{a}\hat{\mu}_i + 2\hat{\lambda}_i),\end{aligned}$$

using

$$\|u\| \le C(N)\|u\|_{H_0^2} \le C(N)\|w_*\|_X,$$

we have

$$\begin{aligned}\|\hat{q}_1 u\| &\le (2\hat{a}^2 + \hat{a}^4 C(N))\|w_*\|_X, \\ \|\hat{q}_2 u\| &\le \left(\hat{a}\hat{R} - \hat{\lambda}_i + (2\hat{a}\hat{R} + \hat{a}^2\hat{\lambda}_i)C(N)\right)\|w_*\|_X, \\ \|\hat{q}_3 u\| &\le \left(\hat{R} - \hat{\mu}_i + (2\hat{R} + \hat{a}(\hat{a}\hat{\mu}_i + 2\hat{\lambda}_i))C(N)\right)\|w_*\|_X, \\ \|\hat{q}_4 u\| &\le (4\hat{a} + 4\hat{a}^3 C(N))\|w_*\|_X.\end{aligned}$$

Then denoting

$$\begin{aligned}s_5 &:= 2\hat{a}^2 + \hat{a}\hat{R} - \hat{\lambda}_i + C(N)\hat{a}\left(\hat{a}^3 + 2\hat{R} + \hat{a}\hat{\lambda}_i\right), \\ s_6 &:= \hat{R} - \hat{\mu}_i + 4\hat{a} + C(N)\left(2\hat{R} + \hat{a}(\hat{a}\hat{\mu}_i + 2\hat{\lambda}_i) + 4\hat{a}^3\right),\end{aligned}$$

we obtain

$$\begin{aligned}\|qw_*\|_Y^2 &\le s_5^2\|w_*\|_X^2 + s_5^2\|w_*\|_X^2 + (s_5+s_6)^2\|w_*\|_X^2 + (s_5+s_6)\|w_*\|_X^2 \\ &= \left(2s_5^2 + 2(s_5+s_6)^2\right)\|w_*\|_X^2.\end{aligned}$$

Consequently ν_3 can be taken as

$$\nu_3 = \sqrt{2}\left(s_5^2 + (s_5+s_6)^2\right)^{1/2}.$$

INTLAB code

“compute_nu3.m”

```
format long;
intvalinit('DisplayInfsup');

K = 200;
C = sqrt(c3(2*K+2));
fprintf('      Constant C(N):%15.5e\n',sup(C))

[a,R,lambda,mu,ur,ui,vr,vi] = input_approximate_solution(K);

s5 = 2*a^2 +  a*R - lambda + C*a*( a^3 + 2*R + a*lambda);
s6 = R - mu +  4*a + C*( 2*R + a*( a*mu + 2*lambda) + 4*a^3);

nu3 = sqrt(intval('2'))*sqrt( s5^2 + (s5+s6)^2 );
fprintf('                nu3:%15.5e\n',sup(nu3))
```

```
>> compute_nu3
===> Default display of intervals by infimum/supremum (e.g. [ 3.14 , 3.15 ])
      Constant C(N):    6.14232e-06
                nu3:    1.27493e+04
>> nu3
intval nu3 =
  1.0e+004 *
[   1.27492957343536,   1.27492957343537]
```